STONE-TYPE DUALITIES FOR SEPARATION LOGICS

SIMON DOCHERTY AND DAVID PYM

University College London, London, UK
e-mail address: simon.docherty.14@ucl.ac.uk

University College London and The Alan Turing Institute, London, UK
e-mail address: d.pym@ucl.ac.uk

Abstract. Stone-type duality theorems, which relate algebraic and relational/topological models, are important tools in logic because — in addition to elegant abstraction — they strengthen soundness and completeness to a categorical equivalence, yielding a framework through which both algebraic and topological methods can be brought to bear on a logic. We give a systematic treatment of Stone-type duality for the structures that interpret bunched logics, starting with the weakest systems, recovering the familiar BI and Boolean BI, and extending to both classical and intuitionistic Separation Logic. The structures we consider are the most general of all known existing algebraic approaches to Separation Logic and thus encompass them all. We demonstrate the uniformity of this analysis by additionally capturing the bunched logics obtained by extending BI and BBI with multiplicative connectives corresponding to disjunction, negation and falsum: De Morgan BI, Classical BI, and the sub-classical family of logics extending Bi-intuitionistic (B)BI. We additionally recover soundness and completeness theorems for the specific truth-functional models of these logics as presented in the literature, with new results given for DMBI, the sub-classical logics extending BiBI and a new bunched logic, CKBI, inspired by the interpretation of Concurrent Separation Logic in concurrent Kleene algebra. This approach synthesises a variety of techniques from modal, substructural and categorical logic and contextualizes the 'resource semantics' interpretation underpinning Separation Logic amongst them. As a consequence, theory from those fields — as well as algebraic and topological methods — can be applied to both Separation Logic and the systems of bunched logics it is built upon. Conversely, the notion of *indexed frame* (generalizing the standard memory model of Separation Logic) and its associated completeness proof can easily be adapted to other non-classical predicate logics.

1. Introduction

1.1. **Background.** Bunched logics, beginning with O'Hearn and Pym's **BI** [60], have proved to be exceptionally useful tools in modelling and reasoning about computational and information-theoretic phenomena such as resources, the structure of complex systems, and access control [19, 20, 32]. Perhaps the most striking example is Separation Logic [62, 67]

Key words and phrases: Algebraic logic, bunched logic, concurrent Kleene algebra, correspondence theory, hyperdoctrine, Kripke semantics, modal logic, non-classical logic, predicate logic, program logic, separation logic, Stone-type duality, substructural logic.

LOGICAL METHODS IN COMPUTER SCIENCE

DOI:10.2168/LMCS-???

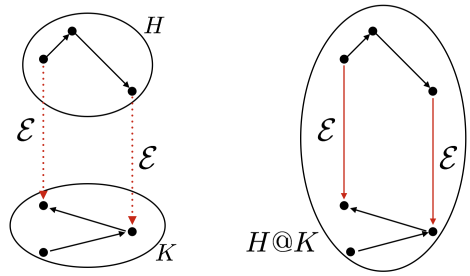

Figure 1: A layered graph $H \,@_{\mathcal{E}}\, K$

(via Pointer Logic [44]), a specific theory of first-order (Boolean) BI with primitives for mutable data structures. Other examples include layered graph logics [19, 20, 32], modal and epistemic systems [27, 38], and Hennessy–Milner-style process logics that have applications in security [20] and systems modelling [2, 21].

The weakest bunched systems are the so-called layered graph logics [19, 32]. These logics have a multiplicative conjunction that is neither associative nor commutative, together with its associated implications, and additives that may be classical or intuitionistic. These systems can be used to describe the decomposition of directed graphs into layers (see Fig 1), with applications such as complex systems modelling (e.g., [19, 32]) and issues in security concerning the relationship of policies and the systems to which they are intended to apply (e.g., [20, 32]). Strengthening the multiplicative conjunction to be associative and commutative and adding a multiplicative unit yields **BI**, for intuitionistic additives, and Boolean BI (**BBI**), for classical additives. Further extensions include additive and multiplicative modalities and, with the addition of parametrization of modalities on actions, Hennessy–Milner-style process logics [21, 2]. Yet further extensions include additive and multiplicative epistemic modalities [38], with applications in security modelling.

All of the applications of bunched logics to reasoning about computational and information-theoretic phenomena essentially rely on the interpretation of the truth-functional models of these systems known as *resource semantics*. Truth-functional models of bunched logics are, essentially, constructed from pre- or partially ordered partial monoids [40] which, in resource semantics, are interpreted as describing how resource-elements can be combined (monoid composition) and compared (order). The program logic known as *Separation Logic* [44, 62, 67] is a specific theory of first-order bunched logic based on the partial monoid of elements of the heap (with the order being simply equality). Separation Logic has found industrial-strength application to static analysis through Facebook's Infer tool (`fbinfer.com`).

Stone's representation theorem for Boolean algebras [64] establishes that every Boolean algebra is isomorphic to a field of sets. Specifically, every Boolean algebra $\mathbb{A}$ is isomorphic to the algebra of clopen subsets of its associated *Stone space* [47] $S(\mathbb{A})$. This result generalizes to a family of Stone-type duality theorems which establish equivalences between certain categories of topological spaces and categories of partially ordered sets. From the logical point of view, Stone-type dualities strengthen the semantic equivalence of truth-functional (such as **BI**'s resource semantics or Kripke's semantics for intuitionistic logic) and algebraic (such as BI algebras or Heyting algebras) models to a dual equivalence of categories. This is useful for a number of reasons: on the one hand, it provides a theoretically convenient abstract characterization of semantic interpretations and, on the other, it provides a systematic approach to soundness and completeness theorems, via the close relationship between the algebraic structures and Hilbert-type proof systems. Beyond this, Stone-type dualities set up a framework through which techniques from both algebra and topology can be brought to bear on a logic.

1.2. **Contributions.** In this paper, we give a systematic account of resource semantics via a family of Stone-type duality theorems that encompass the range of systems from the layered graph logics, via **BI** and **BBI**, to Separation Logic. Our analysis is extended to bunched logics with additional multiplicative connectives [10, 11, 14] and — through straightforward combination with the analogous results from the modal logic literature — can be given for the modal and epistemic systems extending **(B)BI** [26, 27, 38]. As corollaries we retrieve the soundness and completeness of the standard truth-functional models in the literature as well as several new ones.

Soundness and completeness theorems for bunched logics and their extensions tend to be proved through labelled tableaux countermodel procedures [27, 38, 40, 53] that must be specified on a logic-by-logic basis, or by lengthy translations into auxilliary modal logics axiomatized by Sahlqvist formulae [11, 14, 16]. A notable exception to this (and precursor of the completeness result for **(B)BI** given in the present work) is [39]. We predict our framework will increase the ease with which completeness theorems can be proved for future bunched logics, as the family of duality theorems can be extended in a modular fashion.

Of particular interest here are bunched logics with intuitionistic additives. In translations into Sahlqvist-axiomatized modal logics, some connectives must be converted into their 'diamond-like' De Morgan dual and back. For example, magic wand $-\!\!*$ must be encoded as septraction — that is, $\phi \multimap\!\!\circledast\, \psi := \neg(\neg\phi -\!\!* \neg\psi)$ — in a logically equivalent Sahlqvist-axiomatized modal logic. This is not possible on weaker-than-Boolean bases, which necessarily lack negation. Our direct proof method side steps this issue and as a result we are able to give the first completeness proofs for the intuitionistic variants of bunched logics that were not amenable to the proof method used for their Boolean counterparts. We further demonstrate the viability of this proof technique for bunched logics by specifying a new logic **CKBI**, derived from the interpretation of Concurrent Separation Logic in concurrent Kleene algebra, and prove it sound and complete via duality. More generally, the notion of *indexed frame* (generalizing the standard model of Separation Logic) and its associated completeness proof can easily be adapted to other non-classical predicate logics.

All of the structures given in existing algebraic and relational approaches to Separation Logic — including [12], [17], [29], [33], and [34] — are instances of the structures utilized in the present work. Thus these approaches are all proved sound with respect to the standard semantics on store-heap pairs by the results of this paper. In particular, we strengthen Biering et al.'s [5] interpretation of Separation Logic in BI hyperdoctrines to a dual equivalence of categories. To do so we synthesise a variety of related work from modal [48, 66], relevant [1], substructural [6] and categorical logic [24], and thus mathematically some of what follows may be familiar, even if the application to Separation Logic is not. Much of the theory these areas enjoy is produced by way of algebraic and topological arguments. We hope that by recontextualizing the resource semantics of bunched logics in this way similar theory can be given for both Separation Logic and its underlying systems.

1.3. **Structure.** The paper proceeds as described below. In Section 2, we introduce the core bunched logics: the weakest systems **LGL** and **ILGL**, the resource logics **BI** and **BBI**, and finally the program logic Separation Logic (in both classical and intuitionistic form). In Section 3, we define the algebraic and topological structures suitable for interpreting (**I**)**LGL** and give representation and duality theorems relating them. In Section 4 these results are extended to the logics of bunched implications, (**B**)**BI**. In Section 5, we consider categorical structure appropriate for giving algebraic and truth-functional semantics for first-order (B)BI (**FO**(**B**)**BI**). We recall how **FO**(**B**)**BI** can be interpreted on (B)BI hyperdoctrines and define new structures called *indexed (B)BI frames*. Crucially, we show that the standard models of Separation Logic are instantiations of an indexed frames. We show that there is a dual adjunction between (B)BI hyperdoctrines and indexed (B)BI frames given by the semantics and extend (**B**)**BI** duality to obtain a dual equivalence of categories. In Section 6 we extend the theory of the previous sections to a variety of bunched logics extended with additional multiplicative connectives. In doing so we give new completeness theorems for **DMBI** and the logics extending **BiBI**, and specify a new logic **CKBI**. In Section 7, we consider possibilities for further work as a result of the duality theorems.

The present work is an extended and expanded version of a conference paper presented at MFPS XXXIII [30]. New to this version are the corresponding results for the intuitionistic variants of the logics considered there, as well as the extension to bunched logics with additional multiplicatives. The soundness and completeness results for **DMBI** and the various extensions of **BiBI** are, to our knowledge, new: the same is true for **CKBI**, including the specification of the logic itself. Proofs have been included wherever possible.

2. Preliminaries

2.1. **Layered Graph Logics.** We begin by presenting the layered graph logics **LGL** [19] and **ILGL** [32]. First, we give a formal, graph-theoretic definition of layered graph that, we claim, captures the concept as used in modelling complex systems [19, 20, 32]. Informally, two layers in a directed graph are connected by a specified set of edges, each element of which starts in the upper layer and ends in the lower layer.

Given a directed graph, $\mathcal{G}$, we refer to its *vertex set* and its *edge set* by $V(\mathcal{G})$ and $E(\mathcal{G})$ respectively, while its set of subgraphs is denoted $Sg(\mathcal{G})$, with $H \subseteq \mathcal{G}$ iff $H \in Sg(\mathcal{G})$. For a *distinguished edge set* $\mathcal{E} \subseteq E(\mathcal{G})$, the *reachability relation* $\rightsquigarrow_{\mathcal{E}}$ on $Sg(\mathcal{G})$ is defined $H \rightsquigarrow_{\mathcal{E}} K$ iff a vertex of K can be reached from a vertex of H by an $\mathcal{E}$-edge. This generates a partial composition $@_{\mathcal{E}}$ on subgraphs, with $H @_{\mathcal{E}} K \downarrow$ (where $\downarrow$ denotes definedness) iff $V(H) \cap V(K) = \emptyset, H \rightsquigarrow_{\mathcal{E}} K$ and $K \not\rightsquigarrow_{\mathcal{E}} H$. Output is given by the graph union of the two subgraphs and the $\mathcal{E}$-edges between them. We say G is a *layered graph* (with respect to $\mathcal{E}$) if there exist H, K such that $H @_{\mathcal{E}} K \downarrow$ and $G = H @_{\mathcal{E}} K$ (see Fig 1). Layering is evidently neither commutative nor associative.

Let Prop be a set of atomic propositions, ranged over by p. The set of all formulae of **LGL** and **ILGL** is generated by the following grammar:

$$\phi ::= \mathrm{p} \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \phi \blacktriangleright \phi \mid \phi \mathrel{-\!\!\blacktriangleright} \phi \mid \phi \mathrel{\blacktriangleright\!\!-} \phi.$$

The connectives above are the standard logical connectives, together with a (non-commutative and non-associative) multiplicative conjunction, $\blacktriangleright$, and its associated implications $-\!\!\blacktriangleright$ and $\blacktriangleright\!\!-$, in the spirit of the Lambek calculus [51, 52]. We define $\neg\phi$ as $\phi \to \bot$. Hilbert-type

0.	$\dfrac{}{\neg\neg\phi \vdash \phi}$	1.	$\dfrac{}{\phi \vdash \phi}$	2.	$\dfrac{}{\phi \vdash \top}$	
3.	$\dfrac{}{\bot \vdash \phi}$	4.	$\dfrac{\eta \vdash \phi \quad \eta \vdash \psi}{\eta \vdash \phi \wedge \psi}$	5.	$\dfrac{\phi \vdash \psi_1 \wedge \psi_2}{\phi \vdash \psi_i}$	$i = 1, 2$
6.	$\dfrac{\phi \vdash \psi}{\eta \wedge \phi \vdash \psi}$	7.	$\dfrac{\eta \vdash \psi \quad \phi \vdash \psi}{\eta \vee \phi \vdash \psi}$	8.	$\dfrac{\phi \vdash \psi_i}{\phi \vdash \psi_1 \vee \psi_2}$	$i = 1, 2$
9.	$\dfrac{\eta \vdash \phi \to \psi \quad \eta \vdash \phi}{\eta \vdash \psi}$	10.	$\dfrac{\eta \wedge \phi \vdash \psi}{\eta \vdash \phi \to \psi}$	11.	$\dfrac{\xi \vdash \phi \quad \eta \vdash \psi}{\xi \blacktriangleright \eta \vdash \phi \blacktriangleright \psi}$	
12.	$\dfrac{\eta \blacktriangleright \phi \vdash \psi}{\eta \vdash \phi \mathrel{-\!\!\blacktriangleright} \psi}$	13.	$\dfrac{\xi \vdash \phi \mathrel{-\!\!\blacktriangleright} \psi \quad \eta \vdash \phi}{\xi \blacktriangleright \eta \vdash \psi}$	14.	$\dfrac{\eta \blacktriangleright \phi \vdash \psi}{\phi \vdash \eta \mathrel{\blacktriangleright\!\!-} \psi}$	
		15.	$\dfrac{\xi \vdash \phi \mathrel{\blacktriangleright\!\!-} \psi \quad \eta \vdash \phi}{\eta \blacktriangleright \xi \vdash \psi}$			

Figure 2: Hilbert systems for **(I)LGL**: $\mathrm{ILGL_H}$ is given by 1 - 15, $\mathrm{LGL_H}$ by 0–15.

$G \vDash$	p	iff	$G \in \mathcal{V}(\mathrm{p})$	$G \vDash$	$\top$			$G \nvDash \bot$
$G \vDash$	$\phi \wedge \psi$	iff	$G \vDash \phi$ and $G \vDash \psi$	$G \vDash$	$\phi \vee \psi$	iff	$G \vDash \phi$ or	$G \vDash \psi$
$G \vDash$	$\phi \to \psi$	iff	for all $H \succcurlyeq G$, $H \vDash \phi$ implies $H \vDash \psi$					
$G \vDash$	$\phi \blacktriangleright \psi$	iff	there exists H, K such that $H \mathbin{@_{\mathcal{E}}} K \downarrow$, $H \mathbin{@_{\mathcal{E}}} K \preccurlyeq G$, $H \vDash \phi$ and $K \vDash \psi$					
$G \vDash$	$\phi \mathrel{-\!\!\blacktriangleright} \psi$	iff	for all H, K such that $H \mathbin{@_{\mathcal{E}}} K \downarrow$ and $G \preccurlyeq H$, $H \vDash \phi$ implies $H \mathbin{@_{\mathcal{E}}} K \vDash \psi$					
$G \vDash$	$\phi \mathrel{\blacktriangleright\!\!-} \psi$	iff	for all H, K such that $K \mathbin{@_{\mathcal{E}}} H \downarrow$ and $G \preccurlyeq H$, $H \vDash \phi$ implies $K \mathbin{@_{\mathcal{E}}} H \vDash \psi$					

Figure 3: Satisfaction on layered graphs for **(I)LGL**. **LGL** is the case where $\preccurlyeq$ is $=$.

systems for the logics are given in Fig 2: **ILGL** is specified by rules 1–15, whilst **LGL** is specified by rules 0–15.

LGL and **ILGL** are interpreted on *scaffolds*: structures $\mathcal{X} = (\mathcal{G}, \mathcal{E}, X, \preccurlyeq)$ where $\mathcal{G}$ is a directed graph, $\mathcal{E}$ is a distinguished edge set, $X \subseteq Sg(\mathcal{G})$ is such that – if $H \mathbin{@_{\mathcal{E}}} K \downarrow$ – $H, K \in X$ iff $H \mathbin{@_{\mathcal{E}}} K \in X$ and $\preccurlyeq$ is a preorder on X. We note that for any $\mathcal{G}, \mathcal{E}$ and X there are always two canonical orders one can consider: the subgraph and the supergraph relations.

To model the logic soundly in the intuitionistic case, valuations $\mathcal{V} : \mathrm{Prop} \to \mathcal{P}(X)$ (where $\mathcal{P}(X)$ is the power set of X) must be *persistent*: for all $G, H \in X$, if $G \in \mathcal{V}(\mathrm{p})$ and $G \preccurlyeq H$, then $H \in \mathcal{V}(\mathrm{p})$. This has a spatial interpretation when we consider the order to be the subgraph relation: if p designates that a resource is located in the region G, it should also hold of the region H containing G.

Given a scaffold $\mathcal{X}$ and a persistent valuation $\mathcal{V}$ the satisfaction relation $\vDash$ for **ILGL** is inductively defined in Fig 3; the particular case where $\preccurlyeq$ is equality yields **LGL**. As is necessary for a sound interpretation of intuitionistic logic, persistence extends to all formulas with respect to the semantics of Fig 3: for all ϕ, G, H, if $G \vDash \phi$ and $G \preccurlyeq H$ then $H \vDash \phi$. We define validity of ϕ in a model $\mathcal{X}$ to mean for all $G \in X, G \vDash \phi$. Validity of ϕ is defined to be validity in all models. Finally, the entailment relation $\phi \vDash \psi$ holds iff for all models and all G, $G \vDash \phi$ implies $G \vDash \psi$.

11′.	$\dfrac{\xi \vdash \phi \quad \eta \vdash \psi}{\xi * \eta \vdash \phi * \psi}$	12′.	$\dfrac{\eta * \phi \vdash \psi}{\eta \vdash \phi \mathrel{-\!\!*} \psi}$	13′.	$\dfrac{\xi \vdash \phi \mathrel{-\!\!*} \psi \quad \eta \vdash \phi}{\xi * \eta \vdash \psi}$
14′.	$\dfrac{}{(\phi * \psi) * \xi \vdash \phi * (\psi * \xi)}$	15′.	$\dfrac{}{\phi * \psi \vdash \psi * \phi}$	16.	$\dfrac{}{\phi * \mathrm{I} \dashv\vdash \phi}$

Figure 4: Rules for the **(B)BI** Hilbert Systems, $(\mathrm{B})\mathrm{BI}_{\mathrm{H}}$.

$r \vDash$	p	iff	$r \in \mathcal{V}(\mathrm{p})$	$r \vDash$	$\top$		$r \nvDash \bot$
$r \vDash$	$\phi \wedge \psi$	iff	$r \vDash \phi$ and $r \vDash \psi$	$r \vDash$	$\phi \vee \psi$	iff	$r \vDash \phi$ or $r \vDash \psi$
$r \vDash$	$\phi \to \psi$	iff	for all $r' \succcurlyeq r$, $r' \vDash \phi$ implies $r' \vDash \psi$;	$r \vDash$	I	iff	$r \in E$
$r \vDash$	$\phi * \psi$	iff	there exists r', r'' such that $r \in r' \circ r''$, $r' \vDash \phi$ and $r'' \vDash \psi$				
$r \vDash$	$\phi \mathrel{-\!\!*} \psi$	iff	for all r', r'' such that $r'' \in r \circ r'$, $r' \vDash \phi$ implies $r'' \vDash \psi$				

Figure 5: Satisfaction on UDMFs for **(B)BI**. **BBI** is the case where $\preccurlyeq$ is $=$.

2.2. **BI and Boolean BI.** Next we present the resource logics **BI** and **BBI** [60]. Let Prop be a set of atomic propositions, ranged over by p. The set of all formulae of **(B)BI** is generated by the following grammar:

$$\phi ::= \mathrm{p} \mid \top \mid \bot \mid \mathrm{I} \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \phi * \phi \mid \phi \mathrel{-\!\!*} \phi.$$

Once again we have the standard logical connectives, this time joined by multiplicative conjunction $*$ and implication $\mathrel{-\!\!*}$, as well as a constant I. By extending rules 1–10 of Fig 2 with the rules of Fig 4 we obtain a system for **BI**; extending rules 0–10 of Fig 2 with the rules of Fig 4 yields a system for **BBI**. These rules enforce commutativity and associativity of the multiplicative conjunction $*$, the adjointness between $*$ and its associated implication $\mathrel{-\!\!*}$, and the fact that I is a unit for $*$.

Models of these logics arise from the analysis of the abstract notion of resource first outlined in the original work on **BI** [60]. There, the principal properties were determined to be composability and comparison, formalized by a commutative and associative composition $\circ$ with unit e, and a pre- or partial order $\preccurlyeq$. The interplay between composition and comparison in this analysis is formalized by a *bifunctorality* condition: if $a \preccurlyeq a'$ and $b \preccurlyeq b'$ then $a \circ a' \preccurlyeq b \circ b'$. Structures $\mathcal{M} = (M, \preccurlyeq, \circ, e)$ satisfying these conditions are known as *resource monoids*.

(B)BI is interpreted on ordered monoidal structures that generalize the notion of resource monoid in various ways. For example, $\circ$ can be generalized to be non-deterministic and/or partial, the unit e can be generalized to a set of units E, and the bifunctorality condition can be generalized to a number of different coherence conditions between $\circ$ and $\preccurlyeq$. The majority of these choices define the same notion of validity, although there are some sharp boundaries (an analysis of some of these issues can be found in [54]), and all of them can be grouped under the name *resource semantics*. As an example we give a class of models that corresponds closely to the standard intuitionistic model of Separation Logic. This class will be further generalized in Section 4 for the purposes of the duality theorem.

A *monoidal frame* is a structure $\mathcal{X} = (X, \preccurlyeq, \circ, E)$ s.t. $\preccurlyeq$ is a preorder, $\circ : X^2 \to \mathcal{P}(X)$ is commutative operation satisfying *non-deterministic associativity*

$$\forall x, y, z, t, s : s \in x \circ y \text{ and } t \in s \circ z \text{ implies } \exists s'(s' \in y \circ z \text{ and } t \in x \circ s),$$

and E is a set of units satisfying, for all $x, y \in X$, the conditions (Weak) $\exists e \in E(x \in x \circ e)$; (Contr) $x \in y \circ e \wedge e \in E \to y \preccurlyeq x$; and (Up) $e \in E \wedge e \preccurlyeq e' \to e' \in E$. It is *downwards-closed* if, whenever $x \in y \circ z$, $y' \preccurlyeq y$ and $z' \preccurlyeq z$, there exists $x' \preccurlyeq x$ such that $x' \in y' \circ z'$. Conversely it is *upwards-closed* if, whenever $x \in y \circ z$ and $x \preccurlyeq x'$, there exists $y' \succcurlyeq y$ and $z' \succcurlyeq z$ such that $x' \in y' \circ z'$.

The notions of upwards and downwards closure are properties that the majority of models of Separation Logic satisfy. Cao et al. [18] show that when a model satisfies both upwards and downwards closure, the semantic clauses of $*$ and $-\!\!*$ can be given identically in the classical and intuitionistic cases.

Given an upwards and downwards closed monoidal frame (UDMF) $\mathcal{X}$ and a persistent valuation $\mathcal{V} : \mathrm{Prop} \to \mathcal{P}(\mathrm{X})$, the satisfaction relation $\vDash$ is inductively defined in Fig 5. Once again, **BBI** is obtained as the particular case where the order $\preccurlyeq$ is equality $=$. As with **ILGL**, persistence extends to all formulas of **BI**, with upwards and downwards closure ensuring that this is the case for formulas of the form $\phi * \psi$ and $\phi -\!\!* \psi$. Validity and entailment are defined in much the same way as the case for **(I)LGL**.

2.3. **Separation Logic.** Separation Logic [56], introduced by Ishtiaq and O'Hearn [44], and Reynolds [62], is an extension of Hoare's program logic which addresses reasoning about programs that access and mutate data structures. The usual presentation of Separation Logic is based on Hoare triples — for reasoning about the state of imperative programs — of the form $\{\,\phi\,\}\, C\, \{\,\psi\,\}$, where C is a program command, ϕ is pre-condition for C, and ψ is a post-condition for C. The formulas ϕ and ψ are given by the following grammar:

$$\phi ::= E = E' \mid E \mapsto F \mid \top \mid \bot \mid \mathrm{Emp} \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \phi * \phi \mid \phi -\!\!* \phi \mid \exists v.\phi \mid \forall v.\phi.$$

Reynolds' programming language is a simple language of commands with a Lisp-like set-up for creating and accessing cons cells: $C ::= x := E \mid x := E.i \mid E.i := E' \mid x := \mathit{cons}(E_1, E_2) \mid \ldots$. Here the expressions E of the language are built up using booleans, variables, etc., *cons* cells, and atomic expressions. Separation Logic thus facilitates verification procedures for programs that alter the heap.

A key feature of Separation Logic is the local reasoning provided by the so-called Frame Rule,

$$\frac{\{\,\phi\,\}\, C\, \{\,\psi\,\}}{\{\,\phi * \chi\,\}\, C\, \{\,\psi * \chi\,\}},$$

where χ does not include any free variables modified by the program C. Static analysis procedures based on the Frame Rule form the basis of Facebook's Infer tool (`fbinfer.com`) that is deployed in its code production. The decomposition of the analysis that is facilitated by the Frame Rule is critical to the practical deployability of Infer.

Separation Logic can usefully and safely be seen (see [67] for the details) as a presentation of (B)BI Pointer Logic [44]. The semantics of (B)BI Pointer Logic, a theory of (first-order) **(B)BI**, is an instance of **(B)BI**'s resource semantics in which the monoid of resources is constructed from the program's heap. In detail, this model has two components, the store and the heap. The store is a partial function mapping from variables to values $a \in \mathrm{Val}$, such as integers, and the heap is a partial function from natural numbers to values. In logic,

$s, h \vDash E = E'$	iff $\{\{E\}\}s = \{\{E'\}\}s$	$s, h \vDash \top$	$s, h \nvDash \bot$
$s, h \vDash \phi \wedge \psi$	iff $s, h \vDash \phi$ and $s, h \vDash \psi$	$s, h \vDash \phi \vee \psi$	iff $s, h \vDash \phi$ or $s, h \vDash \psi$
$s, h \vDash \phi \to \psi$	iff for all $h' \sqsupseteq h$, $h' \vDash \phi$ implies $h' \vDash \psi$	$s, h \vDash$ Emp	iff $h \sqsupseteq []$
$s, h \vDash \phi * \psi$	iff there exists h', h'' s.t. $h \# h'$, $h = h' \cdot h''$, $s, h' \vDash \phi$ and $s, h'' \vDash \psi$		
$s, h \vDash \phi \mathrel{-\!\!*} \psi$	iff for all h' such that $h \# h', s, h' \vDash \phi$ implies $s, h \cdot h' \vDash \psi$		
$s, h \vDash \exists v.\phi$	iff there exists $a \in \mathrm{Val}$, $[s \mid v \mapsto a], h \vDash \phi$		
$s, h \vDash \forall v.\phi$	iff for all $a \in \mathrm{Val}$, $[s \mid v \mapsto a], h \vDash \phi$		

Figure 6: Satisfaction for (B)BI Pointer Logic. The BBI variant replaces $\sqsupseteq$ with $=$.

the store is often called the valuation, and the heap is a possible world. In programming languages, the store is sometimes called the environment. Within this set-up, the atomic formulae of (B)BI Pointer Logic include equality between expressions, $E = E'$, and, crucially, the points-to predicate, $E \mapsto F$.

We use the following additional notation: $dom(h)$ denotes the domain of definition of a heap h and $dom(s)$ is the domain of a store s; $h \# h'$ denotes that $dom(h) \cap dom(h') = \emptyset$; $h \cdot h'$ denotes the union of functions with disjoint domains, which is undefined if the domains overlap; $h \sqsubseteq h'$ denotes that the graph of h is a subgraph of h'; $[]$ denotes the *empty heap* that is nowhere defined; $[f \mid v \mapsto a]$ is the partial function that is equal to f except that v maps to a; expressions E are built up from variables and constants, and so determine denotations $\{\{E\}\}s \in \mathrm{Val}$.

With this basic data we can define the satisfaction relations for BI and BBI pointer logic. BI pointer logic is given by extending the *intuitionistic* semantic clause for $\mapsto$,

$$s, h \vDash E \mapsto F \text{ iff } \{\{E\}\}s \in dom(h) \text{and } h(\{\{E\}\}s) = \{\{F\}\}s,$$

with those in Fig 6; similarly, BBI pointer logic is given by extending the *classical* semantic clause for $\mapsto$,

$$s, h \vDash E \mapsto F \text{ iff } \{\{E\}\}s = \mathrm{dom}(h) \text{ and } h(\{\{E\}\}s) = \{\{F\}\}s,$$

with those in Fig 6, where $\sqsubseteq$ is replaced for $=$ in the clauses for $\to$ and Emp. The judgement, $s, h \vDash \phi$, says that the assertion ϕ holds for a given store and heap, assuming that the free variables of ϕ are contained in the domain of s. The classical interpretation of $\mapsto$ requires E to be the only active address in the current heap, whereas the intuitionistic interpretation is the weaker judgement that E is *at least* one of the active addresses in the current heap.

The technical reason for this difference is the requirement of persistence with respect to the heap extension ordering $\sqsubseteq$ for soundness of the intuitionistic semantics. This leads to some quirks in the intuitionistic model: for example, the multiplicative unit Emp collapses to $\top$ as every heap extends the empty heap. Some aspects remain the same: as heaps are upwards and downwards closed (in the sense defined in Section 2.2) the semantic clauses for $*$ and $\mathrel{-\!\!*}$ can be given identically. Most importantly, in both cases descriptions of larger heaps can be built up using $*$, and this corresponds to the local reasoning provided by the Frame Rule.

When Separation Logic is given as BI pointer logic it is known as *Intuitionistic Separation Logic*, whereas the formulation as BBI pointer logic is known as *Classical Separation Logic*. Although both were defined in the paper introducing Separation Logic as a theory of bunched logic [44], the classical variant has taken precedence in both theoretical work and practical implementations. A key exception to this is the higher-order Concurrent Separation Logic framework IRIS, which is based on Intuitionistic Separation Logic [49]. IRIS requires an underlying intuitionistic logic as it utilizes the 'later' modality $\triangleright$ [55], which collapses to triviality ($\triangleright\phi \leftrightarrow \top$ for all ϕ) in the presence of the law of excluded middle.

3. Layered Graph Logics

3.1. **Algebra and Frames for (I)LGL.** We begin our analysis with the weakest systems, **LGL** and **ILGL**. Each of the logics we consider can be obtained by extending the basic structures associated with these logics and so we are able to systematically extend the theory in each case by accounting for just the extensions to the structure. First, we consider lattice-based algebras suitable for interpreting (**I**)**LGL**.

Definition 3.1 ((I)LGL Algebra)**.**

(1) An *ILGL algebra* is an algebra $\mathbb{A}$=$(A, \wedge, \vee, \rightarrow, \top, \bot, \bullet, -\!\!\bullet, \bullet\!\!-)$ such that $(A, \wedge, \vee, \rightarrow, \top, \bot)$ is a Heyting algebra and $\bullet, -\!\!\bullet, \bullet\!\!-$ are binary operations on A satisfying, for all $a, b, c \in A$,
$$a \bullet b \leq c \text{ iff } a \leq b -\!\!\bullet c \text{ iff } b \leq a \bullet\!\!- c.$$
(2) A *LGL algebra* is an algebra $\mathbb{A} = (A, \wedge, \vee, \neg, \top, \bot, \bullet, -\!\!\bullet, \bullet\!\!-)$ such that $(A, \wedge, \vee, \neg, \top, \bot)$ is a Boolean algebra and $\bullet, -\!\!\bullet, \bullet\!\!-$ are binary operations on A satisfying, for all $a, b, c \in A$,
$$a \bullet b \leq c \text{ iff } a \leq b -\!\!\bullet c \text{ iff } b \leq a \bullet\!\!- c.$$

□

Residuation of $\bullet$, $-\!\!\bullet$ and $\bullet\!\!-$ with respect to the underlying lattice order entails a number of useful properties that are utilized in what follows.

Proposition 3.2 (cf. [46])**.** *Let $\mathbb{A}$ be an (I)LGL algebra. Then, for all $a, b, a', b' \in A$ and $X, Y \subseteq A$, we have the following:*

(1) *If $a \leq a'$ and $b \leq b'$ then $a \bullet b \leq a' \bullet b'$;*
(2) *If $\bigvee X$ and $\bigvee Y$ exist then $\bigvee_{x \in X, y \in Y} x \bullet y$ exists and $(\bigvee X) \bullet (\bigvee Y) = \bigvee_{x \in X, y \in Y} x \bullet y$;*
(3) *If $a = \bot$ or $b = \bot$ then $a \bullet b = \bot$;*
(4) *If $\bigvee X$ exists then for any $z \in A$, $\bigwedge_{x \in X}(x -\!\!\bullet z)$ and $\bigwedge_{x \in X}(x \bullet\!\!- z)$ exist with*
$$\bigwedge_{x \in X}(x -\!\!\bullet z) = (\bigvee X) -\!\!\bullet z \text{ and } \bigwedge_{x \in X}(x \bullet\!\!- z) = (\bigvee X) \bullet\!\!- z;$$
(5) *If $\bigwedge X$ exists then for any $z \in A$, $\bigwedge_{x \in X}(z -\!\!\bullet x)$ and $\bigwedge_{x \in X}(z \bullet\!\!- x)$ exist with*
$$\bigwedge_{x \in X}(z -\!\!\bullet x) = z -\!\!\bullet (\bigwedge X) \text{ and } \bigwedge_{x \in X}(z \bullet\!\!- x) = z \bullet\!\!- (\bigwedge X); \text{ and}$$
(6) *$a -\!\!\bullet \top = a \bullet\!\!- \top = \bot -\!\!\bullet a = \bot \bullet\!\!- a = \top$.* □

Interpretations of **(I)LGL** on an (I)LGL algebra work as follows: let $\mathcal{V} : \mathrm{Prop} \to A$ be an assignment of propositional variables to elements of the algebra; this is uniquely extended to an interpretation $[\![-]\!]$ of every **ILGL** formula by induction, with $[\![\mathrm{p}]\!] = \mathcal{V}(\mathrm{p})$, $[\![\top]\!] = \top$ and $[\![\bot]\!] = \bot$ as base cases:

$$\begin{array}{llllll} [\![\phi \wedge \psi]\!] & = [\![\phi]\!] \wedge [\![\psi]\!] & [\![\phi \vee \psi]\!] & = [\![\phi]\!] \vee [\![\psi]\!] & [\![\phi \to \psi]\!] & = [\![\phi]\!] \to [\![\psi]\!] \\ [\![\phi \blacktriangleright \psi]\!] & = [\![\phi]\!] \bullet [\![\psi]\!] & [\![\phi \multimap\!\!\blacktriangleright \psi]\!] & = [\![\phi]\!] \multimap\!\!\bullet [\![\psi]\!] & [\![\phi \blacktriangleright\!\!- \psi]\!] & = [\![\phi]\!] \bullet\!\!- [\![\psi]\!]. \end{array}$$

In the case of **LGL**, the interpretation is the same except that $[\![\phi \to \psi]\!] = \neg[\![\phi]\!] \vee [\![\psi]\!]$. An interpretation $[\![-]\!]$ on an (I)LGL algebra $\mathbb{A}$ *satisfies* ϕ iff $[\![\phi]\!] = \top$; ϕ is valid on (I)LGL algebras iff ϕ is satisfied by every interpretation $[\![-]\!]$ on every ILGL algebra $\mathbb{A}$. By constructing a Lindenbaum-Tarski algebra from the Hilbert systems of Fig 2 we obtain the following soundness and completeness theorem for (I)LGL algebras.

Theorem 3.3 (Algebraic Soundness & Completeness)**.** *For all **(I)LGL** formulas ϕ, ψ, $\phi \vdash \psi$ is provable in* (I)$\mathrm{LGL_H}$ *iff, for all (I)LGL algebras $\mathbb{A}$ and all interpretations $[\![-]\!]$ on $\mathbb{A}$, $[\![\phi]\!] \leq [\![\psi]\!]$.* □

Next we generalize the intended model of the logics to a class of relational structures.

Definition 3.4 ((I)LGL Frame)**.** An *ILGL frame* is a triple $\mathcal{X} = (X, \preccurlyeq, \circ)$ where X is a set, $\preccurlyeq$ a preorder on X and $\circ : X^2 \to \mathcal{P}(X)$ a binary operation. An *LGL frame* is an ILGL frame for which the order $\preccurlyeq$ is equality $=$. □

Here $\circ$ is a generalization of Section 2.1's @. We can therefore reconfigure the semantics given in Figure 3 to give a satisfaction relation on (I)LGL frames by straightforward substitutions of $\circ$. Partiality is encoded by the fact $\circ$ has codomain $\mathcal{P}(X)$ and thus $x \circ y = \emptyset$ holds when $\circ$ is undefined for x, y. However, $\circ$ is strictly more general as it does not necessarily satisfy the partial determinism that holds of scaffolds: $G = H \,@_{\mathcal{E}}\, K$ and $G' = H \,@_{\mathcal{E}}\, K$ implies $G = G'$. As $\circ$ is nondeterministic, it can equivalently be seen as a ternary relation, though we maintain the partial function notation to emphasise its interpretation as a composition operator.

(I)LGL frames can be equipped a notion of morphism to obtain categories ILGLFr and LGLFr. As in modal logic, if there is a surjective (I)LGL morphism $g : \mathcal{X} \to \mathcal{X}'$ then any **(I)LGL** formula valid in $\mathcal{X}$ is also valid in $\mathcal{X}'$.

Definition 3.5 (ILGL Morphism)**.** Given ILGL frames $\mathcal{X}$ and $\mathcal{X}'$, an *ILGL morphism* is a map $g : X \to X'$ satisfying

(1) $x \preccurlyeq y$ implies $g(x) \preccurlyeq' g(y)$,
(2) $g(x) \preccurlyeq' y'$ implies there exists $y \in X$ s.t. $x \preccurlyeq y$ and $g(y) = y'$,
(3) $x \in y \circ z$ implies $g(x) \in g(y) \circ' g(z)$,
(4) $w' \preccurlyeq' g(x)$ and $w' \in y' \circ' z'$ implies there exists $w, y, z \in X$ s.t. $w \preccurlyeq x$, $w \in y \circ z$, $y' \preccurlyeq' g(y)$ and $z' \preccurlyeq' g(z)$,
(5) $g(x) \preccurlyeq' w'$ and $z' \in w' \circ' y'$ implies there exists $w, y, z \in X$ s.t. $x \preccurlyeq w$, $z \in w \circ y$, $y' \preccurlyeq' g(y)$ and $g(z) \preccurlyeq' z'$, and
(6) $g(x) \preccurlyeq' w'$ and $z' \in y' \circ' w'$ implies there exists $w, y, z \in X$ s.t. $x \preccurlyeq w$, $z \in y \circ w$, $y' \preccurlyeq' g(y)$ and $g(z) \preccurlyeq' z'$. □

For **LGL**, replacing $\preccurlyeq$ with $=$ collapses the above definition to a simpler and more familiar notion of morphism.

Definition 3.6 (LGL Morphism (cf. [13]))**.** Given LGL frames $\mathcal{X}$ and $\mathcal{X}'$, a *LGL morphism* is a map $g : \mathcal{X} \to \mathcal{X}'$ satisfying

(1) $x \in y \circ z$ implies $g(x) \in g(y) \circ' g(z)$,
(2) $g(x) \in y' \circ' z'$ implies there exists $y, z \in X$ s.t. $x \in y \circ z, g(y) = y'$ and $g(z) = z'$,
(3) $z' \in g(x) \circ' y'$ implies there exists $y, z \in X$ s.t. $z \in x \circ y, g(y) = y'$ and $g(z) = z'$, and
(4) $z' \in y' \circ' g(x)$ implies there exists $y, z \in X$ s.t. $z \in y \circ x, g(y) = y'$ and $g(z) = z'$. □

3.2. **Duality for (I)LGL.** We now give representation and duality theorems for ILGL algebras. As a corollary we obtain the equivalence of the relational semantics to the algebraic semantics, as well as its completeness with respect to the Hilbert system of Fig 2. We first define two transformations that underpin a dual adjunction between the category of (I)LGL algebras and the category of (I)LGL frames.

Definition 3.7 (ILGL Complex Algebra)**.** Given an ILGL frame $\mathcal{X}$, the *complex algebra* of $\mathcal{X}$ is given by $Com^{\text{ILGL}}(\mathcal{X}) = (\mathcal{P}_{\preccurlyeq}(X), \cap, \cup, \Rightarrow, X, \emptyset, \bullet_{\mathcal{X}}, -\!\!\bullet_{\mathcal{X}}, \bullet\!\!-_{\mathcal{X}})$ where

$$\begin{aligned}
\mathcal{P}_{\preccurlyeq}(X) &= \{A \subseteq X \mid \text{ if } a \in A \text{ and } a \preccurlyeq b \text{ then } b \in A\} \\
A \Rightarrow B &= \{x \mid \text{if } x \preccurlyeq x' \text{ and } x' \in A \text{ then } x' \in B\} \\
A \bullet_{\mathcal{X}} B &= \{x \mid \text{there exists } w, y, z \text{ s.t } w \preccurlyeq x, w \in y \circ z, y \in A \text{ and } z \in B\} \\
A -\!\!\bullet_{\mathcal{X}} B &= \{x \mid \text{for all } w, y, z, \text{ if } x \preccurlyeq w, z \in w \circ y \text{ and } y \in A \text{ then } z \in B\} \\
A \bullet\!\!-_{\mathcal{X}} B &= \{x \mid \text{for all } w, y, z, \text{ if } x \preccurlyeq w, z \in y \circ w \text{ and } y \in A \text{ then } z \in B\}.
\end{aligned}$$

□

Each operator here maps upwards-closed sets to upwards-closed sets so this is well defined. By substituting $\preccurlyeq$ for $=$, the above definition collapses to one suitable for **LGL**.

Definition 3.8 (LGL Complex Algebra)**.** Given a LGL frame $\mathcal{X}$, the *complex algebra* of $\mathcal{X}$ is given by $Com^{\text{LGL}}(\mathcal{X}) = (\mathcal{P}(X), \cap, \cup, \setminus, X, \emptyset, \bullet_{\mathcal{X}}, -\!\!\bullet_{\mathcal{X}}, \bullet\!\!-_{\mathcal{X}})$, where $\bullet_{\mathcal{X}}$, $-\!\!\bullet_{\mathcal{X}}$, and $\bullet\!\!-_{\mathcal{X}}$ are defined as follows:

$$\begin{aligned}
A \bullet_{\mathcal{X}} B &= \{x \mid \text{there exists } y \in A, z \in B \text{ s.t. } x \in y \circ z\} \\
A -\!\!\bullet_{\mathcal{X}} B &= \{x \mid \text{for all } y, z \in X, z \in x \circ y \text{ and } y \in A \text{ implies } z \in B\} \\
A \bullet\!\!-_{\mathcal{X}} B &= \{x \mid \text{for all } y, z \in X, z \in y \circ x \text{ and } y \in A \text{ implies } z \in B\}.
\end{aligned}$$

□

The residuated structure in each case is then easy to verify.

Lemma 3.9. *Given an (I)LGL frame $\mathcal{X}$, $Com^{\text{(I)LGL}}(\mathcal{X})$ is an (I)LGL algebra.* □

Any valuation $\mathcal{V}$ on an (I)LGL frame $\mathcal{X}$ generates an interpretation $[\![-]\!]_{\mathcal{V}}$ on the complex algebra $Com^{\text{(I)LGL}}(\mathcal{X})$. A straightforward inductive argument then shows satisfiability coincides on these two models.

Proposition 3.10. *For any relational (I)LGL model $(\mathcal{X}, \mathcal{V})$, $x \vDash \phi$ iff $x \in [\![\phi]\!]_{\mathcal{V}}$.* □

Conversely we can create ILGL frames from ILGL algebras. We first recall some definitions. A *filter* on a bounded distributive lattice $\mathbb{A}$ is a non-empty set $F \subseteq A$ such that, for all $x, y \in A$, (i) $x \in F$ and $x \leq y$ implies $y \in F$, and (ii) $x, y \in F$ implies $x \wedge y \in F$. It is a *proper* filter if it additionally satisfies (iii) $\bot \notin F$. It is *prime* if in addition it satisfies (iv) $x \vee y \in F$ implies $x \in F$ or $y \in F$. If $\mathbb{A}$ is a Boolean algebra and F a proper filter, (iv) is equivalent to (v) $x \in F$ or $\neg x \in F$ and vi) F is a maximal proper filter with respect to $\subseteq$. The dual notion of a filter is an *ideal*, with proper and prime ideals defined as one would expect. Importantly, if I is a prime ideal then the complement $\overline{I}$ is a prime filter.

Definition 3.11 (Prime Filter (I)LGL Frame)**.** Given an ILGL algebra $\mathbb{A}$, the *prime filter frame* of $\mathbb{A}$ is given by $Pr^{\text{ILGL}}(\mathbb{A}) = (Pr(A), \subseteq, \circ_{\mathbb{A}})$ where

$$F \circ_{\mathbb{A}} F' = \{F'' \mid \forall a \in F, \forall b \in F' : a \bullet b \in F''\}.$$

For a LGL algebra $\mathbb{A}$, the prime filter frame is simply $Pr^{\text{LGL}}(\mathbb{A}) = (Pr(A), \circ_{\mathbb{A}})$. □

Lemma 3.12. *Given an (I)LGL algebra $\mathbb{A}$, $Pr^{\text{(I)LGL}}(\mathbb{A})$ is an ILGL frame.* □

In analogy with Stone's representation theorem for Boolean algebras, we can give a representation theorem for (I)LGL algebras using these constructions. In particular, for ILGL algebras this extends the representation theorem for Heyting algebras [36], whereas for LGL algebras this extends Stone's theorem. These results are closely related to various representation theorems for algebras with operators (e.g., [48], [43]). The key difference is the use of a single operation $\circ$ for the operator $\bullet$ *and* its non-operator adjoints $-\!\!\bullet$ and $\bullet\!\!-$. The derived structure required to take care of these adjoints was not investigated in the frameworks of Jonsson-Tarski or Goldblatt but has been in the context of gaggle theory [6, 35]. There the result for LGL algebras can be found as a particular case of that for Boolean gaggles ([6], Theorem 1.4.16).

Theorem 3.13 (Representation Theorem for (I)LGL Algebras)**.** *Every (I)LGL algebra is isomorphic to a subalgebra of a complex algebra. Specifically, given an (I)LGL algebra $\mathbb{A}$, the map $\theta_{\mathbb{A}} : \mathbb{A} \to Com^{\text{(I)LGL}}(Pr^{\text{(I)LGL}}(\mathbb{A}))$ defined $\theta_{\mathbb{A}}(a) = \{F \in Pr^{\text{(I)LGL}}(\mathbb{A}) \mid a \in F\}$ is an embedding.*

Proof. We prove the theorem for ILGL algebras as the case for **LGL** is then an immediate corollary obtained by substituting $\preccurlyeq$ for $=$ throughout. That $\theta_{\mathbb{A}}$ is injective and respects the Heyting algebra operations is simply the representation theorem for Heyting algebras. It thus remains to show $\theta_{\mathbb{A}}$ respects the operations $\bullet$, $-\!\!\bullet$ and $\bullet\!\!-$. We address $-\!\!\bullet$ and leave the (similar) remaining cases to the reader.

We first note the simple boundary cases: for all $a, b \in A$ we trivially have that $\theta_{\mathbb{A}}(a -\!\!\bullet \top) = \theta_{\mathbb{A}}(a) -\!\!\bullet_{Pr(\mathbb{A})} \theta_{\mathbb{A}}(\top)$ and $\theta_{\mathbb{A}}(\bot -\!\!\bullet b) = \theta_{\mathbb{A}}(\bot) -\!\!\bullet_{Pr(\mathbb{A})} \theta_{\mathbb{A}}(b)$ by Proposition 3.2 (6). Hence it is sufficient to consider $a -\!\!\bullet b$ where $a \neq \bot$ and $b \neq \top$. First suppose $a -\!\!\bullet b \neq \bot$. For the inclusion $\theta_{\mathbb{A}}(a -\!\!\bullet b) \subseteq \theta_{\mathbb{A}}(a) -\!\!\bullet_{Pr(\mathbb{A})} \theta_{\mathbb{A}}(b)$, assume $a -\!\!\bullet b \in F$ with F_0, F_1, F_2 such that $F \subseteq F_0$, $F_2 \in F_0 \circ_{\mathbb{A}} F_1$ and $a \in F_1$. Then $(a -\!\!\bullet b) \bullet a \in F_2$ and so $b \in F_2$ since residuation entails $(a -\!\!\bullet b) \bullet b \leq b$ and F_2 is upwards closed. Hence $F \in \theta_{\mathbb{A}}(a) -\!\!\bullet_{Pr(\mathbb{A})} \theta_{\mathbb{A}}(b)$.

Conversely, assume $a -\!\!\bullet b \notin F$. We show prime filters F_0 and F_1 exist such that $F_2 \in F \circ_{\mathbb{A}} F_1$ with $a \in F_1$ but $b \notin F_2$. Consider the sets $f_1 = \{x \mid a \leq x\}$ and $i_2 = \{x \mid x \leq b\}$. As $a \neq \bot$ and $b \neq \top$ these define a proper filter and a proper ideal respectively. Abusing notation, we can see $f_2 \in F \circ_{\mathbb{A}} f_1$, where f_2 is the complement of i_2: if $x \in F$ and $y \geq a$ then $x \bullet y \not\leq b$, otherwise by residuation and monotonicity of $\bullet$ we would have $x \leq a -\!\!\bullet b \in F$, a contradiction. Clearly $a \in F_1$ and $b \in i_2$. By Zorn's Lemma, we can extend f_1 and i_2 to a prime filter F_1 and prime ideal I_2 with these properties (to show that these are prime requires the properties of Proposition 3.2: see [31]). Taking F_2 to be the complement of I_2, itself a prime filter, we have witnesses to the fact $F \notin \theta_{\mathbb{A}}(a) -\!\!\bullet_{Pr(\mathbb{A})} \theta_{\mathbb{A}}(b)$. This also gives us the required inclusion in the case $a -\!\!\bullet b = \bot$ and so we are done. □

That $\theta_{\mathbb{A}}$ is an embedding immediately gives us an analogous result to Prop 3.10.

Corollary 3.14. *For all (I)LGL algebras $\mathbb{A}$: given an interpretation $[\![-]\!]$, the valuation $\mathcal{V}_{[\![-]\!]}(p) = \theta_{\mathbb{A}}([\![p]\!])$ on $Pr^{\text{(I)LGL}}(\mathbb{A})$ is such that $[\![\phi]\!] \in F$ iff $F \vDash \phi$.* □

To strengthen the representation theorem to a dual adjunction we define maps $\eta_{\mathcal{X}}$ for each (I)LGL frame $\mathcal{X}$ by $\eta_{\mathcal{X}}(x) = \{A \in Com^{\text{(I)(LGL)}}(\mathcal{X}) \mid x \in A\}$. By extending $Pr^{\text{(I)LGL}}$ and $Com^{\text{(I)LGL}}$ to contravariant functors by setting $Pr^{\text{(I)LGL}}(f) = f^{-1}$ and $Com^{\text{(I)LGL}}(g) = g^{-1}$ — that these define the appropriate morphisms is straightforward but tedious — we can see that $\theta_{(-)}$ and $\eta_{(-)}$ in fact define natural transformations that form the dual adjunction.

Theorem 3.15. *The functors $Pr^{(I)LGL}$ and $Com^{(I)LGL}$ and the natural transformations θ and η form a dual adjunction of categories between* (I)LGLAlg *and* (I)LGLFr*.* □

Corollary 3.16 (Relational Soundness and Completeness)**.** *For all formulas ϕ, ψ of **(I)LGL**: $\phi \vdash \psi$ is provable in* (I)LGL_{H} *iff $\phi \vDash \psi$ in the relational semantics.* □

This dual adjunction can be specialized to a dual equivalence of categories by introducing topology to (I)LGL frames. We first define ILGL spaces. This definition (necessarily) extends that of the topological duals of Heyting algebra given by Esakia duality [37]. The coherence conditions on the composition $\circ$ are inspired by those found on the topological duals of gaggles [6].

Definition 3.17 (ILGL Space)**.** An *ILGL space* is a structure $\mathcal{X} = (X, \mathcal{O}, \preccurlyeq, \circ)$ such that:
(1) $(X, \mathcal{O}, \preccurlyeq)$ is an Esakia space [37];
(2) $(X, \preccurlyeq, \circ)$ is an ILGL frame;
(3) The upwards-closed clopen sets of $(X, \mathcal{O}, \preccurlyeq)$ are closed under $\bullet_{\mathcal{X}}, -\!\!\bullet_{\mathcal{X}}, \bullet\!\!-_{\mathcal{X}}$;
(4) If $x \notin y \circ z$ then there exist upwards-closed clopen sets C_1, C_2 such that $y \in C_1, z \in C_2$ and $x \notin C_1 \bullet_{\mathcal{X}} C_2$.

A morphism of ILGL spaces is a continuous ILGL morphism, yielding a category ILGLSp. □

Once again, substituting $\preccurlyeq$ for $=$ in the definition of ILGL space obtains the topological duals for LGL algebras.

Definition 3.18 (LGL Space)**.** An *LGL space* is a structure $\mathcal{X} = (X, \mathcal{O}, \circ)$ such that
(1) $(X, \mathcal{O})$ is an Stone space [64];
(2) $(X, \circ)$ is an LGL frame;
(3) The clopen sets of $(X, \mathcal{O}, \preccurlyeq)$ are closed under $\bullet_{\mathcal{X}}, -\!\!\bullet_{\mathcal{X}}, \bullet\!\!-_{\mathcal{X}}$;
(4) If $x \notin y \circ z$ then there exist clopen sets C_1, C_2 such that $y \in C_1, z \in C_2$ and $x \notin C_1 \bullet_{\mathcal{X}} C_2$.

A morphism of LGL spaces is a continuous LGL morphism, yielding a category ILGLSp. □

We now adapt $Pr^{\text{(I)LGL}}, Com^{\text{(I)LGL}}, \theta$ and η for these subcategories. First, it is straightforward to augment the prime filter frame with the topological structure required to make it an (I)LGL space. We define a subbase by $S = \{\theta_{\mathbb{A}}(a) \mid a \in A\} \cup \{\overline{\theta_{\mathbb{A}}(a)} \mid a \in A\}$ where $\overline{\theta_{\mathbb{A}}(a)}$ denotes the set complement. This generates a topology $\mathcal{O}_{\mathbb{A}}$ and we can define $PrSp^{\text{ILGL}}$: ILGLAlg $\to$ ILGLSp by $PrSp^{\text{LGL}}(\mathbb{A}) = (Pr(A), \mathcal{O}_{\mathbb{A}}, \subseteq, \circ_{\mathbb{A}})$ and $PrSp^{\text{LGL}}(f) = f^{-1}$. In the case of **LGL**, the sets $\overline{\theta_{\mathbb{A}}(a)}$ are redundant as for every prime filter F on a Boolean algebra $\mathbb{A}$, $a \in A$ implies $a \in F$ or $\neg a \in F$. Hence $\mathcal{B} = \{\theta_{\mathbb{A}}(a) \mid a \in A\}$ defines a base for a topology $\mathcal{O}_{\mathbb{A}}$ and we can set $PrSp^{\text{LGL}}(\mathbb{A}) = (Pr(A), \mathcal{O}_{\mathbb{A}}, \circ_{\mathbb{A}})$ and $PrSp^{\text{LGL}}(f) = f^{-1}$.

Conversely, given an ILGL space $\mathcal{X}$ we now take the set of upwards-closed clopen sets $\mathcal{CL}_{\preccurlyeq}(\mathcal{X})$ as the carrier of an ILGL algebra, together with the operations of the ILGL complex algebra; ILGL space property (3) ensures this is well defined. Hence we take $Clop^{\text{ILGL}}$: ILGLSp $\to$ ILGLAlg to be $Clop^{\text{ILGL}}(\mathcal{X}) = (\mathcal{CL}_{\preccurlyeq}(\mathcal{X}), \cap, \cup, \Rightarrow, X, \emptyset, \bullet_{\mathcal{X}}, -\!\!\bullet_{\mathcal{X}}, \bullet\!\!-_{\mathcal{X}})$ and $Clop^{\text{ILGL}}(g) = g^{-1}$. For **LGL** we simply take the clopen sets of the underlying topological space $\mathcal{CL}(\mathcal{X})$) together with the operations of the LGL complex algebra. Hence $Clop^{\text{LGL}}(\mathcal{X}) = (\mathcal{CL}(\mathcal{X}), \cap, \cup, \setminus, X, \emptyset, \bullet_{\mathcal{X}}, \multimap\!\!\bullet_{\mathcal{X}}, \bullet\!\!-_{\mathcal{X}})$ and $Clop^{\text{LGL}}(g) = g^{-1}$.

The well-definedness of these functors can be seen by straightforwardly combining Stone (Esakia) duality for Boolean (Heyting) algebras with the results relating to the dual adjunction of (I)LGL structures. Relativizing θ and η to these new functors — explicitly, $\theta_{\mathbb{A}}(a) = \{F \in Pr(A) \mid a \in F\}$ and $\eta_{\mathcal{X}}(x) = \{A \in Clop^{\text{(I)LGL}}(\mathcal{X}) \mid x \in A\}$ — renders them natural isomorphisms. As with the functors, this is mostly a consequence of Esakia duality and the arguments relating to the dual adjunction. However, in order to show each $\eta_{\mathcal{X}}$ is an isomorphism on the (I)LGL frame structure we must use (I)LGL space property (4). We thus obtain the duality theorem. For LGL algebras this is also obtainable as a specific case of Bimbó & Dunn's duality theorem for Boolean gaggles ([6], Theorem 9.2.22).

Theorem 3.19 (Duality Theorem for (I)LGL)**.** *The categories of (I)LGL algebras and (I)LGL spaces are dually equivalent.* □

4. The Logics of Bunched Implications

4.1. **Algebra and Frames for (B)BI.** We now extend the results of the previous section to the logics **BI** and **BBI** by systematically extending the structures defined for **LGL** and **ILGL**. We begin once again with the lattice-based algebras that interpret the logics.

Definition 4.1 ((B)BI Algebra)**.**
(1) A *BI algebra* is an ILGL algebra $\mathbb{A}$ extended with a constant I such that $(A, \bullet, \mathrm{I})$ is a commutative monoid.
(2) A *BBI algebra* is a LGL algebra $\mathbb{A}$ extended with a constant I such that $(A, \bullet, \mathrm{I})$ is a commutative monoid. □

Thus BI algebras are the subclass of ILGL algebras with a commutative and associative $\bullet$ that has unit I, and likewise for BBI algebras with respect to LGL algebras. In particular, commutativity of $\bullet$, together with residuation, causes $\multimap\!\!\bullet = \bullet\!\!-$. Interpretations on a BI and BBI algebras are given in much the same way as **ILGL** and **LGL**, except $*$ and $-\!\!*$ are interpreted by $\bullet$ and $\multimap\!\!\bullet$ — $[\![\phi * \psi]\!] = [\![\phi]\!] \bullet [\![\psi]\!]$ and $[\![\phi -\!\!* \psi]\!] = [\![\phi]\!] \multimap\!\!\bullet [\![\psi]\!]$ — and $[\![\mathrm{I}]\!] = \mathrm{I}$. A soundness and completeness theorem for algebraic interpretations is proved in the same fashion as Theorem 3.3.

Theorem 4.2 (Algebraic Soundness & Completeness)**.** *For* ***(B)BI*** *formulas ϕ, ψ, $\phi \vdash \psi$ is provable in* (B)BI$_{\mathrm{H}}$ *iff for all (B)BI algebras $\mathbb{A}$ and all interpretations $[\![-]\!]$ on $\mathbb{A}$, $[\![\phi]\!] \leq [\![\psi]\!]$.* □

Next we define a class of relational models appropriate for interpreting **(B)BI**. The outermost universal quantification in each frame condition is left implicit for readability.

Definition 4.3 (BI Frame)**.** A *BI Frame* is a tuple $\mathcal{X} = (X, \preccurlyeq, \circ, E)$ such that $\preccurlyeq$ a preorder on X, $\circ : X^2 \to \mathcal{P}(X)$ is a binary operation, and $E \subseteq X$ a set, satisfying

$$\begin{array}{rlrr}
\text{(Comm)} & z \in x \circ y \to z \in y \circ x & \text{(Up)} & e \in E \wedge e \preccurlyeq e' \to e' \in E \\
\text{(Weak)} & \exists e \in E(x \in x \circ e) & \text{(Contr)} & x \in y \circ e \wedge e \in E \to y \preccurlyeq x \\
\text{(Assoc)} & \multicolumn{3}{l}{t' \succcurlyeq t \in x \circ y \wedge w \in t' \circ z \to \exists s, s', w'(s' \succcurlyeq s \in y \circ z \wedge w \succcurlyeq w' \in x \circ s').}
\end{array}$$

□

The notion of BI frame and its associated semantics can be defined in many ways, and indeed those given in Section 2.2 can suffice for a duality theorem if one defines the natural transformations and morphisms slightly differently. These choices are driven by the requirements that the model validates the associativity axiom and that the semantic clauses for $*$ and $-\!\!*$ satisfy persistence. This requires careful interplay between the definition of (Assoc), the conditions relating $\circ$ and $\preccurlyeq$, and the semantic clauses for $*$ and $-\!\!*$. One such solution is that given in the original papers on BI [60], in which $\circ$ is a deterministic, associative function that is bifunctorial with respsect to the order. This is complicated somewhat when $\circ$ is non-deterministic and partial, however. A general analysis of the choices available for such a $\circ$ when the definition of the associativity-like condition it satisfies is kept fixed can be found in [18].

Here, we use a definition that enables a uniform extension of the structures and theorems relating to **ILGL**. In particular, we use a more general than usual formulation of (Assoc) that allows us to directly extend the results of Section 3. This solution places no coherence conditions on $\circ$ and $\preccurlyeq$ and thus requires what Cao et al. [18] call the "strong semantics" for $*$ and $-\!\!*$ to maintain persistence. These clauses are those given for **ILGL**'s multiplicatives. Explicitly, we have

$$\begin{array}{lcl} x \vDash \phi * \psi & \text{iff} & \text{there exists } x', y, z \text{ s.t. } x' \preccurlyeq x, x' \in y \circ z, y \vDash \phi \text{ and } z \vDash \psi \\ x \vDash \phi -\!\!* \psi & \text{iff} & \text{for all } x', y, z \text{ s.t. } x \preccurlyeq x' \text{ and } z \in x' \circ y, y \vDash \phi \text{ implies } z \vDash \psi. \end{array}$$

This can be seen as a strict generalization of the model given in Section 2.2, as witnessed by the fact that the definition of associativity given there implies (Assoc) when $\circ$ is additionally upwards and downwards closed.

Proposition 4.4. *Every upwards and downwards closed monoidal frame is a BI frame.* □

Further, the respective semantic clauses for $*$ and $-\!\!*$ are equivalent when the underlying model is upwards and downwards closed [18]. The converse does not hold: not every BI frame is upwards or downwards closed. However, every BI frame generates a upwards and downwards closed monoidal frame with an equivalent satisfaction relation. Given a BI frame $\mathcal{X} = (X, \preccurlyeq, \circ, E)$, define its upwards and downwards closure by $\mathcal{X}^{\Uparrow\Downarrow} = (X, \preccurlyeq, \circ^{\Uparrow\Downarrow}, E)$ where $x \in y \circ^{\Uparrow\Downarrow} z$ iff there exist x', y', z' such that $x' \preccurlyeq x$, $y \preccurlyeq y'$, $z \preccurlyeq z'$ and $x' \in y' \circ z'$. By taking care over the respective associativity properties of each frame, this can easily be seen to be an upwards and downwards closed monoidal frame. Letting $\vDash'$ denote the satisfaction relation defined in Fig 5, we have the following result.

Proposition 4.5. *For all BI frames $\mathcal{X}$, persistent valuations $\mathcal{V}$ and formulas ϕ of **BI**, $\mathcal{X}, x \vDash \phi$ iff $\mathcal{X}^{\Uparrow\Downarrow}, x \vDash' \phi$.* □

So, as far as the logic is concerned, these choices are academic: they all collapse to the same notion of validity. For models satisfying one or both of upwards and downwards closure this is implicit in the preservation results of Cao et al. [18]. Our analysis of the remaining class of models that lack both conditions on $\circ$ and $\preccurlyeq$ completes this picture.

Definition 4.6. A *BBI frame* $\mathcal{X}$ is a triple $\mathcal{X} = (X, \circ, E)$, such that $\circ : X^2 \to \mathcal{P}(X)$ is a binary operation and $E \subseteq X$, satisfying

$$\begin{array}{llll} \text{(Comm)} & z \in x \circ y \to z \in y \circ x & \text{(Assoc)} & t \in x \circ y \wedge w \in t \circ z \to \exists s (s \in y \circ z \wedge w \in x \circ s) \\ \text{(Weak)} & \exists e \in E (x \in x \circ e) & \text{(Contr)} & x \in y \circ e \wedge e \in E \to y = x. \end{array}$$

□

We note that when $\preccurlyeq$ is substituted for $=$, all sound choices of the (Assoc) axiom that were possible for BI frames collapse to the axiom given here, while every coherence condition on $\circ$ and $\preccurlyeq$ becomes trivial. Thus, in comparison to **BI**, there are far fewer choices to be made about **BBI** models and so this definition is more familiar, appearing in the literature in precisely the same form as *BBI frames* [13] and *non-deterministic monoids* [54] and slightly modified as *multi-unit separation algebras* [33] and *relational frames* [39].

The key difference with the latter definitions is that multi-unit separation algebras are cancellative — $z \in x \circ y$ and $z \in x \circ y'$ implies $y = y'$ — and relational frames have a single unit. BBI frames do not require cancellativity and have multiple units. This difference is crucial for the present work as the duality theorems do not hold when we restrict to frames satisfying either of these properties. This is witnessed by the fact that **BBI** is not expressive enough to distinguish between cancellative/non-cancellative models and single unit/multi-unit models [13], all of which define the same notion of validity [54]. An interpretation of the duality theorem might thus be that BBI frames are the most general relational structures that soundly and completely interpret **BBI**.

Definition 4.7 ((B)BI Morphism)**.** Given (B)BI frames $\mathcal{X}$ and $\mathcal{X}'$, a (B)BI morphism is an ILGL (LGL) morphism $g : \mathcal{X} \to \mathcal{X}'$ that additionally satisfies (7) $e \in E$ iff $g(e) \in E'$. □

BI frames together with BI morphisms form a category BIFr, itself a subcategory of ILGLFr; likewise, BBI frames and BBI morphisms form the category BBIFr, a subcategory of LGLFr. Note that commutativity of $\circ$ collapses the final conditions in the definition of (I)LGL morphism when defined on BBI (BI) frames. As in the case for the layered graph logics, surjective (B)BI morphisms preserve validity in models.

We now relate the two categories of **(B)BI** model via a dual adjunction.

Definition 4.8 ((B)BI Complex Algebra)**.** Given a (B)BI frame $\mathcal{X}$, the complex algebra of $\mathcal{X}$, $Com^{\text{(B)BI}}(\mathcal{X})$ is given by extending $Com^{\text{ILGL}}(\mathcal{X})$ ($Com^{\text{LGL}}(\mathcal{X})$) with the unit set of $\mathcal{X}$, E. □

Lemma 4.9. *Given a (B)BI frame $\mathcal{X}$, $Com^{(B)BI}(\mathcal{X})$ is a (B)BI algebra.*

Proof. We give the argument for BI frames; the case for BBI frames is obtained from the specific case where $\preccurlyeq$ is $=$ throughout. Given Lemma 3.9, we just need to verify that $(\mathcal{P}_{\preccurlyeq}(X), \bullet_{\mathcal{X}}, E)$ is a commutative monoid. First note that $E \in \mathcal{P}_{\preccurlyeq}(X)$ by virtue of (Up). Commutativity is easily derived from the property (Comm). Further, the inclusion $A \subseteq A \bullet_{\mathcal{X}} E$ follows immediately from (Weak). Slightly more involved is $A \bullet_{\mathcal{X}} E \subseteq A$: let $x \in A \bullet_{\mathcal{X}} E$. Then there exists x', y, e such that $x' \preccurlyeq x$ with $x' \in y \circ e$, $y \in A$ and $e \in E$. By (Contr) it follows that $y \preccurlyeq x'$, so by transitivity $y \preccurlyeq x$. By upwards-closure of A, $x \in A$.

We finally come to associativity. We only need to verify the inclusion $(A \bullet_{\mathcal{X}} B) \bullet_{\mathcal{X}} C \subseteq A \bullet_{\mathcal{X}} (B \bullet_{\mathcal{X}} C)$ because of commutativity of $\bullet_{\mathcal{X}}$. Let $a \in (A \bullet_{\mathcal{X}} B) \bullet_{\mathcal{X}} C$. Then there exists w, t', z such that $a \succcurlyeq w \in t' \circ z$ with $t' \in A \bullet_{\mathcal{X}} B$ and $z \in C$. This entails the existence of x, y, t such that $t' \succcurlyeq t \in x \circ y$ with $x \in A$ and $y \in B$. By (Assoc) we thus have s, s', w' with $s' \succcurlyeq s \in y \circ z$ and $w \succcurlyeq w' \in x \circ s'$. Hence $s' \in B \bullet_{\mathcal{X}} C$ and, because $w' \preccurlyeq w \preccurlyeq a$, $a \in A \bullet_{\mathcal{X}} (B \bullet_{\mathcal{X}} C)$ as required. □

As a special case of the analogous result for **(I)LGL**, we obtain a correspondence between satisfiability on a frame and its complex algebra when the algebraic interpretation is generated by the valuation on the frame.

Proposition 4.10. *For any (B)BI frame $\mathcal{X}$ and valuation $\mathcal{V}$, $x \vDash \phi$ iff $x \in [\![\phi]\!]_{\mathcal{V}}$.* □

In the other direction we transform BI algebras into BI frames.

Definition 4.11 (Prime Filter (B)BI Frame)**.** Given a (B)BI algebra $\mathbb{A}$, the prime filter frame of $\mathbb{A}$, $Pr^{\text{(B)BI}}(\mathbb{A})$, is given by extending $Pr^{\text{ILGL}}(\mathbb{A})$ ($Pr^{\text{LGL}}(\mathbb{A})$) with $E_{\mathbb{A}} = \{F \in Pr(A) \mid \mathrm{I} \in F\}$. □

That the prime filter frame of a (B)BI algebra is a (B)BI frame follows an argument essentially given in the completeness theorem for the relational semantics of **BBI** [39].

Lemma 4.12. *Given a (B)BI algebra $\mathbb{A}$, the prime filter frame $Pr^{(B)BI}(\mathbb{A})$ is a (B)BI frame.*

Proof. In both cases, that $\circ_{\mathbb{A}}$ satisfies (Comm) can be read off of the definition of $\circ_{\mathbb{A}}$ and the commutativity of $*$. For BI algebras, $E_{\mathbb{A}}$ satisfying (Up) is trivial. We are left to prove (Assoc), (Weak) and (Contr). Our argument works for both BI and BBI algebras except for a final step on (Contr).

For (Assoc), suppose $F_{t'} \supseteq F_t \in F_x \circ F_y$ and $F_w \in F_{t'} \circ F_z$. Consider the set

$$f_s = \{a \in A \mid \exists b \in F_y, c \in F_z : a \geq b * c\}.$$

We show that f_s is a proper filter. First suppose $\bot \in f_s$. Then there exist $b \in F_y$, $c \in F_z$ such that $b * c = \bot$. However, if this was the case, for arbitrary $a \in F_x$ we would have $a * b \in F_{t'} \subseteq F_t$,, and so $(a * b) * c = a * (b * c) = a * \bot = \bot \in F_w$, contradicting that F_w is a prime filter. It is easy to see that f_s is upwards-closed so it remains to verify that it is closed under meets. If $a, a' \in f_s$ then there are $b, b' \in F_y$ and $c, c' \in F_z$ such that $a \geq b * c$ and $a' \geq b' * c'$. We have $b \wedge b' \in F_y$ and $c \wedge c' \in F_z$ since they are filters, and, using the monotonicity of $*$ and the fact that $\wedge$ gives greatest lower bounds, it can be shown that $(b \wedge b') * (c \wedge c') \leq (b * c) \wedge (b' * c') \leq a \wedge a'$, so $a \wedge a' \in f_s$.

Abusing notation, it's easy to see that $f_s \in F_y \circ_{\mathbb{A}} F_z$. We also have $F_w \in F_x \circ_{\mathbb{A}} f_s$. Let $a \in F_x$ and $a' \in f_s$. Then there exist $b \in F_y$ and $c \in F_z$ such that $a' \geq b * c$ and $a * a' \geq a * (b * c) = (a * b) * c$. We have $a * b \in F_t \subseteq F_{t'}$ and so $(a * b) * c \in F_w$. Since f_s is a proper filter, we can use Zorn's Lemma to extend f_s to a prime filter F_s satisfying these properties. Then we have $F_s \supseteq F_s \in F_y \circ_{\mathbb{A}} F_z$ and $F_w \supseteq F_w \in F_x \circ_{\mathbb{A}} F_s$ as required.

For (Weak), consider the set $f_e = \{b \in A \mid b \geq \mathrm{I}\}$. We have $F \in F \circ_{\mathbb{A}} f_e$ as, for $a \in F$ and $b \geq \mathrm{I}$, we have $a * b \geq a * \mathrm{I} = a$ so by upwards closure of F, $a * b \in F$. Using Zorn's Lemma we can extend f_e to a prime filter F_e satisfying this property, and so for all $F \in Pr(A)$ we have $F_e \in E_{\mathbb{A}}$ such that $F \in F \circ_{\mathbb{A}} F_e$ as required.

For (Contr) note that if $F_x \in F_y \circ F_e$ and $F_e \in E_{\mathbb{A}}$ we have $\mathrm{I} \in F_e$ and so for all $a \in F_y$, $a * \mathrm{I} = a \in F_x$. Hence $F_y \subseteq F_x$. This suffices for **BI**: however, for **BBI** there is one final step. By invoking the maximality of prime filters on Boolean algebras we may conclude $F_y = F_x$, thus satisfying the BBI frame condition (Contr). □

The representation theorem for (B)BI algebras now follows immediately from the analogous result for ILGL algebras and the fact that $\theta_{\mathbb{A}}(\mathrm{I}) = E_{\mathbb{A}}$.

Theorem 4.13 (Representation Theorem for (B)BI Algebras)**.** *Every (B)BI algebra is isomorphic to a subalgebra of a complex algebra. Specifically, given a (B)BI algebra $\mathbb{A}$, the map $\theta_{\mathbb{A}} : \mathbb{A} \to Com^{(B)BI}(Pr^{(B)BI}(\mathbb{A}))$ defined $\theta_{\mathbb{A}}(a) = \{F \in Pr^{(B)BI}(A) \mid a \in F\}$ is an embedding.* □

Corollary 4.14. *For all (B)BI algebras $\mathbb{A}$, given an interpretation $[\![-]\!]$, the valuation $\mathcal{V}_{[\![-]\!]}(p) = \theta_{\mathbb{A}}([\![p]\!])$ on $Pr^{(B)BI}(\mathbb{A})$ is such that $[\![\phi]\!] \in F$ iff $F \vDash \phi$.* □

Once again $Pr^{\text{(B)BI}}$ and $Com^{\text{(B)BI}}$ can be made into functors by setting $Pr^{\text{(B)BI}}(f) = f^{-1}$ and $Com^{\text{(B)BI}}(g) = g^{-1}$. Then the natural transformation $\eta_{(-)}$ defined for **(I)LGL** also suffices for the dual adjunction for **BI** (**BBI**) when restricted to the appropriate domain.

Theorem 4.15. *The functors $Pr^{(B)BI}$ and $Com^{(B)BI}$ and the natural transformations θ and η form a dual adjunction of categories between* (B)BIAlg *and* (B)BIFr. □

Corollary 4.16. *For all formulas ϕ, ψ of **BI**, $\phi \vdash \psi$ is provable in* $\mathrm{BI_H}$ *iff $\phi \vDash \psi$ in the relational semantics.* □

To obtain a dual equivalence of categories we add topological structure to BI frames. This can be achieved by straightforwardly extending (I)LGL spaces with BBI (BI) frame structure, as well as a coherence condition for the unit set E.

Definition 4.17 (BI Space)**.** A BI space is a structure $\mathcal{X} = (X, \mathcal{O}, \preccurlyeq, \circ, E)$ such that
(1) $(X, \mathcal{O}, \preccurlyeq, \circ)$ is an ILGL space,
(2) $(X, \preccurlyeq, \circ, E)$ is a BI frame, and
(3) E is clopen in $(X, \mathcal{O})$.
A morphism of BI spaces is a continuous BI morphism, yielding a category BISp. □

Definition 4.18 (BBI Space)**.** A *BBI space* is a structure $\mathcal{X} = (X, \mathcal{O}, \circ, E)$ such that
(1) $(X, \mathcal{O}, \circ)$ is an LGL space,
(2) $(X, \circ, E)$ is a BBI frame, and
(3) E is clopen in $(X, \mathcal{O})$.
A morphism of BBI spaces is a continuous BBI morphism, yielding a category BBISp. □

The duality theorems for **BI** and **BBI** follow essentially immediately from that for **ILGL** and **LGL**. The only additional structure that needs to be taken care of is the constant I and the unit set E. We define the functors and natural isomorphisms explicitly for their use in the Separation Logic duality. Hence for **BI** we have $PrSp^{\text{BI}} : \text{BIAlg} \to \text{BISp}$ defined by $PrSp^{\text{BI}}(\mathbb{A}) = (Pr(A), \mathcal{O}_{\mathbb{A}}, \subseteq, \circ_{\mathbb{A}}, E_{\mathbb{A}})$ (where $\mathcal{O}_{\mathbb{A}}$ is as defined for **ILGL**) and once again $PrSp^{\text{BI}}(f) = f^{-1}$; correspondingly, $Clop^{\text{BI}} : \text{BISp} \to \text{BIAlg}$ is given by $Clop^{\text{BI}}(\mathcal{X}) = (\mathcal{CL}_{\preccurlyeq}(\mathcal{X}), \cap, \cup, \Rightarrow, X, \emptyset, \bullet_{\mathcal{X}}, -\!\!\bullet_{\mathcal{X}}, E)$ (where $\bullet_{\mathcal{X}}, -\!\!\bullet_{\mathcal{X}}$ are the ILGL complex algebra operations) and, as in the case for **ILGL**, $Clop^{\text{BI}}(g) = g^{-1}$; θ and η are given precisely as they are in **ILGL** duality, relativized to BIAlg and BISp.

Similarly, for **BBI** we have $PrSp^{\text{BBI}}(\mathbb{A}) = (Pr(A), \mathcal{O}_{\mathbb{A}}, \circ_{\mathbb{A}}, E_{\mathbb{A}})$ (where $\mathcal{O}_{\mathbb{A}}$ is as defined for **LGL**) and $PrSp^{\text{BBI}}(f) = f^{-1}$; $Clop^{\text{BBI}}(\mathcal{X}) = (\mathcal{CL}(\mathcal{X}), \cap, \cup, \setminus, X, \emptyset, \bullet_{\mathcal{X}}, -\!\!\bullet_{\mathcal{X}}, E)$ (where $\bullet_{\mathcal{X}}, -\!\!\bullet_{\mathcal{X}}$ are the LGL complex algebra operations) and $Clop^{\text{BBI}}(g) = g^{-1}$; θ and η are given precisely as they are in **LGL** duality, relativized to BBIAlg and BBISp.

That $E_{\mathbb{A}}$ is clopen in each instance can be seen by the fact that $E = \theta_{\mathbb{A}}(\mathrm{I})$. By Esakia duality, every set of the form $\theta_{\mathbb{A}}(a)$ for some $a \in A$ is an upwards-closed clopen set of the prime filter space of $\mathbb{A}$. Similarly, the clopen sets of the prime filter space of a Boolean algebra $\mathbb{A}$ are the sets $\theta_{\mathbb{A}}(a)$ for $a \in A$ by Stone duality, so the analogous property for BBI spaces holds too. It is also easy to see that η is isomorphic with respect to E. The duality theorems thus obtain.

Theorem 4.19 (Duality Theorem for **(B)BI**)**.** *The categories of (B)BI algebras and (B)BI spaces are dually equivalent.* □

5. Separation Logic

5.1. **Hyperdoctrines and Indexed Frames for Separation Logic.** We now extend the duality theorems for BI and BBI algebras to the algebraic and relational structures suitable for interpreting Separation Logic. First, we must consider first-order (B)BI (**FO(B)BI**). Hilbert-type proof systems $\mathrm{FO(B)BI_H}$ are obtained by extending those given for **(B)BI** in Section 2 with the usual rules for quantifiers (see, e.g., [65]). Second, to give the semantics for the additional structure of **FO(B)BI**, we must expand our definitions from the propositional case with category-theoretic structure. As these semantic structures support it, we consider a many-sorted first-order logic. We start on the algebraic side with BI hyperdoctrines.

Definition 5.1 ((B)BI Hyperdoctrine (cf. [5]))**.** A *(B)BI hyperdoctrine* is a tuple

$$(\mathbb{P} : \mathrm{C}^{op} \to \mathrm{Poset}, (=_X)_{X\in Ob(\mathrm{C})}, (\exists X_\Gamma, \forall X_\Gamma)_{\Gamma,X\in Ob(\mathrm{C})})$$

such that:

(1) C is a category with finite products;
(2) $\mathbb{P} : \mathrm{C}^{op} \to \mathrm{Poset}$ is a functor such that, for each object X in C, $\mathbb{P}(X)$ is a (B)BI algebra, and, for each morphism f in C, $\mathbb{P}(f)$ is a (B)BI algebra homomorphism;
(3) For each object X in C and each diagonal morphism $\Delta_X : X \to X \times X$ in C, the element $=_X \in \mathbb{P}(X \times X)$ is adjoint at $\top_{\mathbb{P}(X)}$. That is, for all $a \in \mathbb{P}(X \times X)$,
$$\top_{\mathbb{P}(X)} \leq \mathbb{P}(\Delta_X)(a) \text{ iff } =_X \leq a\,;$$
(4) For each pair of objects Γ, X in C and each projection $\pi_{\Gamma,X} : \Gamma \times X \to \Gamma$ in C, $\exists X_\Gamma$ and $\forall X_\Gamma$ are left and right adjoint to $\mathbb{P}(\pi_{\Gamma,X})$. That is, they are monotone maps $\exists X_\Gamma : \mathbb{P}(\Gamma \times X) \to \mathbb{P}(\Gamma)$ and $\forall X_\Gamma : \mathbb{P}(\Gamma \times X) \to \mathbb{P}(\Gamma)$ such that, for all $a, b \in \mathbb{P}(\Gamma)$,
$$\begin{array}{lll} \exists X_\Gamma(a) \leq b & \text{iff} & a \leq \mathbb{P}(\pi_{\Gamma,X})(b) \text{ and} \\ \mathbb{P}(\pi_{\Gamma,X})(b) \leq a & \text{iff} & b \leq \forall X_\Gamma(a). \end{array}$$
This assignment of adjoints is additionally natural in Γ: given a morphism $s : \Gamma \to \Gamma'$, the following diagrams commute:

$$\begin{tikzcd} \mathbb{P}(\Gamma' \times X) \arrow[r, "\mathbb{P}(s\times id_X)"] \arrow[d, "\exists X_{\Gamma'}"'] & \mathbb{P}(\Gamma \times X) \arrow[d, "\exists X_\Gamma"] \\ \mathbb{P}(\Gamma') \arrow[r, "\mathbb{P}(s)"'] & \mathbb{P}(\Gamma) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{P}(\Gamma' \times X) \arrow[r, "\mathbb{P}(s\times id_X)"] \arrow[d, "\forall X_{\Gamma'}"'] & \mathbb{P}(\Gamma \times X) \arrow[d, "\forall X_\Gamma"] \\ \mathbb{P}(\Gamma') \arrow[r, "\mathbb{P}(s)"'] & \mathbb{P}(\Gamma) \end{tikzcd}$$

□

(B)BI hyperdoctrines were first formulated by Biering et al. [5] to prove the existence of models of higher-order variants of Separation Logic. There it was shown that the standard model of Separation Logic could be seen as a BBI hyperdoctrine, and thus safely extended with additional structure in the domain C^{op} to directly define higher-order constructs like lists, trees, finite sets and relations inside the logic. The present work strengthens this result to a dual equivalence of categories. Other algebraic models of Separation Logic, like those based on Boolean quantales [29] or formal power series [34], can be seen as particular instantiations of BBI hyperdoctrines.

To specify an interpretation $[\![-]\!]$ of **FOBI** in a BI hyperdoctrine $\mathbb{P}$ we assign each type X an object $[\![X]\!]$ of C, and for each context $\Gamma = \{v_1 : X_1, \ldots, v_n : X_n\}$ we have $[\![\Gamma]\!] = [\![X_1]\!] \times \cdots \times [\![X_n]\!]$. Each function symbol $f : X_1 \times \cdots X_n \to X$ is assigned a morphism

$[\![f]\!] : [\![X_1]\!] \times \cdots [\![X_n]\!] \to [\![X]\!]$. This allows us to inductively assign to every term of type X in context Γ a morphism $[\![t]\!] : [\![\Gamma]\!] \to [\![X]\!]$ in the standard way (see [61]). We additionally assign, for each m-ary predicate symbol P of type $X_1, \ldots, X_m$, $[\![P]\!] \in \mathbb{P}([\![X_1]\!] \times \cdots \times [\![X_m]\!])$. Then the structure of the hyperdoctrine allows us to extend $[\![-]\!]$ to **FOBI** formulae ϕ in context Γ as follows:

$$\begin{array}{llll}
[\![Pt_1 \ldots t_m]\!] = \mathbb{P}(\langle [\![t_1]\!], \ldots, [\![t_m]\!]\rangle)([\![P]\!]) & [\![\phi \wedge \psi]\!] = [\![\phi]\!] \wedge_{\mathbb{P}([\![\Gamma]\!])} [\![\psi]\!] & [\![\top]\!] = \top_{\mathbb{P}([\![\Gamma]\!])} \\
[\![t =_X t']\!] = \mathbb{P}(\langle [\![t]\!], [\![t']\!]\rangle)(=_{[\![X]\!]}) & [\![\phi \vee \psi]\!] = [\![\phi]\!] \vee_{\mathbb{P}([\![\Gamma]\!])} [\![\psi]\!] & [\![\bot]\!] = \bot_{\mathbb{P}([\![\Gamma]\!])} \\
[\![\phi \to \psi]\!] = [\![\phi]\!] \to_{\mathbb{P}([\![\Gamma]\!])} [\![\psi]\!] & [\![\phi * \psi]\!] = [\![\phi]\!] \bullet_{\mathbb{P}([\![\Gamma]\!])} [\![\psi]\!] & [\![\mathrm{I}]\!] = \mathrm{I}_{\mathbb{P}([\![\Gamma]\!])} \\
[\![\phi \mathrel{-\!\!*} \psi]\!] = [\![\phi]\!] \mathrel{-\!\!\bullet}_{\mathbb{P}([\![\Gamma]\!])} [\![\psi]\!] & [\![\exists v : X.\phi]\!] = \exists [\![X]\!]_{[\![\Gamma]\!]}([\![\phi]\!]) & [\![\forall v : X.\phi]\!] = \forall [\![X]\!]_{[\![\Gamma]\!]}([\![\phi]\!]).
\end{array}$$

Substitution of terms is given by $[\![\phi(t/x)]\!] = \mathbb{P}([\![t]\!])([\![\phi]\!])$. ϕ is satisfied by an interpretation $[\![-]\!]$ if $[\![\phi]\!] = \top_{\mathbb{P}([\![\Gamma]\!])}$. ϕ is valid if it is satisfied by all interpretations. For **FOBBI** on BBI hyperdoctrines everything works the same way except $[\![\phi \to \psi]\!] = \neg_{\mathbb{P}([\![\Gamma]\!])}[\![\phi]\!] \vee_{\mathbb{P}([\![\Gamma]\!])} [\![\psi]\!]$. A standard Lindenbaum-Tarski style construction is sufficient to prove soundness and completeness in both cases.

Theorem 5.2. [61, 5] *For all* ***FO(B)BI*** *formulas ϕ, ψ in context Γ, $\phi \vdash^{\Gamma} \psi$ is provable in* $\mathrm{FO(B)BI_H}$ *iff, for all (B)BI hyperdoctrines $\mathbb{P}$ and all interpretations $[\![-]\!]$, $[\![\phi]\!] \leq_{\mathbb{P}([\![\Gamma]\!])} [\![\psi]\!]$.* □

For the other side of the dual adjunction, we introduce new structures: *indexed (B)BI frames*. This definition is adapted from the notion of indexed Stone space presented by Coumans [24] as a topological dual for Boolean hyperdoctrines. In contrast to the duality presented there, we prove the duality for the more general intuitionistic case and additionally consider (typed) equality and universal quantification. They also appear to have some relation to a more general formulation of Shirasu's *metaframes* [63], another type of indexed frame introduced to interpret predicate superintuitionistic and modal logics, but we defer an investigation of this connection to another occasion.

Definition 5.3. An *indexed (B)BI frame* is a functor $\mathcal{R} : \mathrm{C} \to \mathrm{BIFr}$ such that
(1) C is a category with finite products;
(2) For all objects Γ, Γ' and X in C, all morphisms $s : \Gamma \to \Gamma'$ and all product projections $\pi_{\Gamma,X}$, for the following commutative square

$$\begin{tikzcd}
\mathcal{R}(\Gamma \times X) \arrow[r, "\mathcal{R}(\pi_{\Gamma,X})"] \arrow[d, "\mathcal{R}(s \times id_X)"] & \mathcal{R}(\Gamma) \arrow[d, "\mathcal{R}(s)"'] \\
\mathcal{R}(\Gamma' \times X) \arrow[r, "\mathcal{R}(\pi_{\Gamma',X})"'] & \mathcal{R}(\Gamma')
\end{tikzcd}$$

   (a) (for indexed BI frames) the (Pseudo Epi) property holds: $\mathcal{R}(\pi_{\Gamma',X})(y) \preccurlyeq \mathcal{R}(s)(x)$ implies there exists z such that: $\mathcal{R}(\pi_{\Gamma,X})(z) \preccurlyeq x$ and $y \preccurlyeq \mathcal{R}(s \times id_X)(z)$;
   (b) (for indexed BBI frames) the quasi-pullback property holds: the induced map $\mathcal{R}(\Gamma \times X) \to \mathcal{R}(\Gamma) \times_{\mathcal{R}(\Gamma')} \mathcal{R}(\Gamma' \times X)$ is an epimorphism.

Given an arbitrary indexed BI frame $\mathcal{R} : \mathrm{C} \to \mathrm{(B)BIFr}$ and an object X we denote the BI frame at X by $\mathcal{R}(X) = (\mathcal{R}(X), \preccurlyeq_{\mathcal{R}(X)}, \circ_{\mathcal{R}(X)}, E_{\mathcal{R}(X)})$. Analogously, we denote the BBI frame at X by $\mathcal{R}(X) = (\mathcal{R}(X), \circ_{\mathcal{R}(X)}, E_{\mathcal{R}(X)})$ in the case of an indexed BBI frame. □

$x, [\![-]\!] \vDash^\Gamma$	$Pt_1 \ldots t_m$	iff $\mathcal{R}(\langle [\![t_1]\!], \ldots, [\![t_m]\!] \rangle)(x) \in [\![P]\!]$	$x, [\![-]\!] \vDash^\Gamma \top$
$x, [\![-]\!] \vDash^\Gamma$	$t =_X t'$	iff $\mathcal{R}(\langle [\![t]\!], [\![t']\!] \rangle)(x) \in Ran(\mathcal{R}(\Delta_{[\![X]\!]}))$	$x, [\![-]\!] \nvDash^\Gamma \bot$
$x, [\![-]\!] \vDash^\Gamma$	$\phi \wedge \psi$	iff $x, [\![-]\!] \vDash^\Gamma \phi$ and $x, [\![-]\!] \vDash^\Gamma \psi$	
$x, [\![-]\!] \vDash^\Gamma$	$\phi \vee \psi$	iff $x, [\![-]\!] \vDash^\Gamma \phi$ or $x, [\![-]\!] \vDash^\Gamma \psi$	
$x, [\![-]\!] \vDash^\Gamma$	$\phi \to \psi$	iff for all $x' \succcurlyeq_{\mathcal{R}([\![\Gamma]\!])} x$, $x', [\![-]\!] \vDash^\Gamma \phi$ implies $x', [\![-]\!] \vDash^\Gamma \psi$	
$x, [\![-]\!] \vDash^\Gamma$	I	iff $x \in E_{\mathcal{R}([\![\Gamma]\!])}$	
$x, [\![-]\!] \vDash^\Gamma$	$\phi * \psi$	iff there exists $x' \preccurlyeq x$ s.t. $x' \in y \circ_{\mathcal{R}([\![\Gamma]\!])} z$, $y, [\![-]\!] \vDash^\Gamma \phi$ and $z, [\![-]\!] \vDash^\Gamma \psi$	
$x, [\![-]\!] \vDash^\Gamma$	$\phi \mathrel{-\!\!*} \psi$	iff for all $x' \succcurlyeq x$ s.t. $z \in x' \circ_{\mathcal{R}([\![\Gamma]\!])} y$, $y, [\![-]\!] \vDash^\Gamma \phi$ implies $z, [\![-]\!] \vDash^\Gamma \psi$	
$x, [\![-]\!] \vDash^\Gamma$	$\exists v_{n+1} : X\phi$	iff there exists $x' \in \mathcal{R}([\![\Gamma]\!] \times [\![X]\!])$ s.t. $\mathcal{R}(\pi_{[\![\Gamma]\!],[\![X]\!]})(x') = x$ and $x', [\![-]\!] \vDash^{\Gamma \cup \{v_{n+1}:X\}} \phi$	
$x, [\![-]\!] \vDash^\Gamma$	$\forall v_{n+1} : X\phi$	iff for all $x' \in \mathcal{R}([\![\Gamma]\!] \times [\![X]\!])$, $\mathcal{R}(\pi_{[\![\Gamma]\!],[\![X]\!]})(x') \succcurlyeq_{\mathcal{R}([\![\Gamma]\!])} x$, implies $x', [\![-]\!] \vDash^{\Gamma \cup \{v_{n+1}:X\}} \phi$	

Figure 7: Satisfaction on indexed (B)BI frames for **FO(B)BI**. **FOBBI** replaces $\preccurlyeq$ with $=$.

A truth-functional semantics can be given for **FO(B)BI** on indexed (B)BI frames. For **FOBI**, an interpretation $[\![-]\!]$ is given in precisely the same way as for BI hyperdoctrines, except for the key-difference that each m-ary predicate symbol P of type $X_1, \ldots, X_m$ is assigned to an upwards closed subset $[\![P]\!] \in \mathcal{P}_{\preccurlyeq}(\mathcal{R}([\![X_1]\!] \times \cdots \times [\![X_m]\!]))$. Similarly, an interpretation $[\![-]\!]$ for **FOBBI** is given in the same way as it is for BBI hyperdoctrines, except that, for every m-ary predicate symbol P of type $X_1, \ldots, X_m$, P is assigned to a subset $[\![P]\!] \in \mathcal{P}(\mathcal{R}([\![X_1]\!] \times \cdots \times [\![X_m]\!]))$. Then for formulas ϕ of **FO(B)BI** in context Γ with $x \in \mathcal{R}([\![\Gamma]\!])$ the satisfaction relation $\vDash^\Gamma$ is inductively defined in Fig 7. There, $Ran(\mathcal{R}(\Delta_{[\![X]\!]})) = \{y \mid \exists z(\mathcal{R}(\Delta_{[\![X]\!]})(z) = y)\}$. We note that bound variables are renamed to be fresh throughout, in an order determined by quantifier depth.

The familiar persistence property of propositional intuitionistic logics also holds for satisfaction on indexed BI frames. For atomic predicate formulas this is by design, with the assignment of predicate symbols to upwards closed subsets akin to a persistent valuation. For formulas of the form $t =_X t'$ this follows from the fact that $\mathcal{R}(\Delta_X)$ is a BI morphism and hence order preserving. The rest of the clauses follow by an inductive argument, the most involved of which is for formulas of the form $\exists v_{n+1} : X\phi$. Suppose $x \vDash \mathcal{R}(\pi_{[\![\Gamma]\!],[\![X]\!]})$ and $x \preccurlyeq_{\mathcal{R}([\![\Gamma]\!])} y$. Then there is x' such that $\mathcal{R}(\pi_{[\![\Gamma]\!],[\![X]\!]})(x') = x \preccurlyeq_{\mathcal{R}([\![\Gamma]\!])} y$ and $x', [\![-]\!] \vDash^{\Gamma \cup \{v_{n+1}:X\}} \phi$. Since $\mathcal{R}(\pi_{[\![\Gamma]\!],[\![X]\!]})$ is a BI morphism it satisfies property (2) of Definition 3.5: there exists y' such that $x' \preccurlyeq_{\mathcal{R}([\![\Gamma]\!] \times [\![X]\!])} y'$ and $\mathcal{R}(\pi_{[\![\Gamma]\!],[\![X]\!]})(y') = y$. By inductive hypothesis, $y', [\![-]\!] \vDash^{\Gamma \cup \{v_{n+1}:X\}} \phi$ and so $y, [\![-]\!] \vDash^{\Gamma \cup \{v_{n+1}:X\}} \phi$.

5.2. **Pointer Logic as an Indexed Frame.** Although at first sight it may not seem so, indexed frames and the semantics based upon them are a generalization of the standard store–heap semantics of Separation Logic.

Consider the BI frame $\mathrm{Heap}^{\mathrm{BI}} = (H, \uplus, \sqsubseteq, H)$, where H is the set of heaps, $\sqsubseteq$ is heap extension, and $\uplus$ is defined by $h_2 \in h_0 \uplus h_1$ iff $h_0 \# h_1$ and $h_0 \cdot h_1 = h_2$. This is the BI frame corresponding to the partial monoid of heaps. We define an indexed BI frame $\mathrm{Store}^{\mathrm{BI}} : \mathrm{Set} \to \mathrm{BIFr}$ on objects by $\mathrm{Store}^{\mathrm{BI}}(X) = (X \times H, \uplus_X, \sqsubseteq_X, X \times H)$, where

$(x_2, h_2) \in (x_0, h_0) \uplus_X (x_1, h_1)$ iff $x_0 = x_1 = x_2$ and $h_2 \in h_0 \uplus h_1$, and $(x_0, h_0) \sqsubseteq_X (x_1, h_1)$ iff $x_0 = x_1$ and $h_0 \sqsubseteq h_1$. On morphisms, set $\mathrm{Store}^{\mathrm{BI}}(f : X \to Y)(x, h) = (f(x), h)$. It is straightforward to see this defines a functor: for arbitrary X, $\mathrm{Store}(X)$ inherits the BI frame properties from Heap and for arbitrary $f : X \to Y$, $\mathrm{Store}(f)$ is trivially a BI morphism as it is identity on the structure that determines the back and forth conditions. The property (Pseudo Epi) is also trivially satisfied so this defines an indexed BI frame.

For BBI pointer logic, we instead start with the BBI frame $\mathrm{Heap}^{\mathrm{BBI}} = (H, \uplus, \{[]\})$ where $[]$ is the empty heap. Then $\mathrm{Store}^{\mathrm{BBI}}$ is defined in essentially the same way, with $\mathrm{Store}^{\mathrm{BBI}}(X) = (X \times H, \uplus_X, X \times \{[]\})$ and $\mathrm{Store}^{\mathrm{BBI}}(f)(x, h) = (f(x), h)$. This defines an indexed BBI frame.

We now describe the interpretations $[\![-]\!]$ on $\mathrm{Store}^{\mathrm{(B)BI}}$ that yield the standard models of Separation Logic. We have one type Val and we set $[\![\mathrm{Val}]\!] = \mathbb{Z}$, with the arithmetic operations $[\![+]\!], [\![-]\!] : [\![\mathrm{Val}]\!]^2 \to [\![\mathrm{Val}]\!]$ defined as one would expect. Term morphisms $[\![t]\!] : [\![\mathrm{Val}]\!]^n \to [\![\mathrm{Val}]\!]$ in context $\Gamma = \{v_1, \ldots v_n\}$ are then defined as usual, with each constant n assigned the morphism $[\![n]\!] : [\![\Gamma]\!] \longrightarrow \{*\} \xrightarrow{\ n\ } [\![\mathrm{Val}]\!]$. As one would expect, the key difference between the two interpretations is in the interpretation of the points-to predicate. For Intuitionistic Separation Logic, the points-to predicate $\mapsto$ is assigned

$$[\![\mapsto]\!] = \{((a, a'), h) \mid a \in dom(h) \text{ and } h(a) = a'\} \in \mathcal{P}_{\sqsubseteq_{[\![\mathrm{Val}]\!]^2}}(\mathrm{Store}^{\mathrm{BI}}([\![\mathrm{Val}]\!]^2)).$$

This set is clearly upwards closed with respect to the order $\sqsubseteq_{[\![\mathrm{Val}]\!]^2}$ so this is a well-defined interpretation. For Classical Separation Logic, $\mapsto$ is instead assigned

$$[\![\mapsto]\!] = \{((a, a'), h) \mid \{a\} = dom(h) \text{ and } h(a) = a'\} \in \mathcal{P}(\mathrm{Store}^{\mathrm{BBI}}([\![\mathrm{Val}]\!]^2)).$$

In the indexed (B)BI frame $\mathrm{Store}^{\mathrm{(B)BI}} : \mathrm{Set} \to \mathrm{ResFr}$ with the interpretations just defined, a store is represented as an n-place vector of values over $[\![\mathrm{Val}]\!]$. That is, the store $s = \{(v_1, a_1), \ldots, (v_n, a_n)\}$ is given by the element $(a_1, \ldots, a_n) \in [\![\mathrm{Val}]\!]^n$. By a simple inductive argument we have the following result:

Theorem 5.4. *For all formulas ϕ of (B)BI pointer logic, all stores $s = \{(v_1, a_1), \ldots, (v_n, a_n)\}$ and all heaps h, $s, h \vDash \phi$ iff $((a_1, \ldots, a_n), h), [\![-]\!] \vDash^{\Gamma} \phi$.* □

After verifying that terms are evaluated to the same elements as the standard model in both representations, the equivalence of the clauses for atomic formulas can be computed directly. That the clauses for the multiplicatives $*$ and $-\!*$ are equivalent is a consequence of the upwards and downwards closure of heap composition with respect to heap extension, as discussed in Section 4. Finally, the equivalence of the quantifier clauses is down to the representation of stores as vectors and the action of the product projections under the functor Store. The notions of indexed (B)BI frame and its associated semantics are therefore a natural generalization of the standard Separation Logic model.

5.3. **Duality for (B)BI Hyperdoctrines.** We now extend the results given for (B)BI algebras to (B)BI hyperdoctrines. For such results to make sense, both (B)BI hyperdoctrines and indexed (B)BI frames need to be equipped with a notion of morphism to form categories. Our definition of hyperdoctrine morphism adapts that for *coherent* hyperdoctrines [25].

Definition 5.5 ((B)BI Hyperdoctrine Morphism)**.** Given a pair of (B)BI hyperdoctrines $\mathbb{P} : \mathrm{C}^{op} \to \mathrm{Poset}$ and $\mathbb{P}' : \mathrm{D}^{op} \to \mathrm{Poset}$, a *(B)BI hyperdoctrine morphism* $(\mathrm{K}, \tau) : \mathbb{P} \to \mathbb{P}'$ is a pair (K, τ) satisfying the following properties:

(1) K : C $\to$ D is a finite product preserving functor;
(2) $\tau : \mathbb{P} \to \mathbb{P}' \circ K$ is a natural transformation;
(3) For all objects X in C: $\tau_{X \times X}(=_X) = \; ='_{K(X)}$;
(4) For all objects Γ and X in C, the following squares commute:

$$\begin{array}{ccc} \mathbb{P}(\Gamma \times X) & \xrightarrow{\tau_{\Gamma\times X}} & \mathbb{P}'(K(\Gamma) \times K(X)) \\ \downarrow{\scriptstyle \exists X_\Gamma} & & \downarrow{\scriptstyle \exists' K(X)_{K(\Gamma)}} \\ \mathbb{P}(\Gamma) & \xrightarrow{\tau_\Gamma} & \mathbb{P}'(K(\Gamma)) \end{array} \qquad \begin{array}{ccc} \mathbb{P}(\Gamma \times X) & \xrightarrow{\tau_{\Gamma\times X}} & \mathbb{P}'(K(\Gamma) \times K(X)) \\ \downarrow{\scriptstyle \forall X_\Gamma} & & \downarrow{\scriptstyle \forall' K(X)_{K(\Gamma)}} \\ \mathbb{P}(\Gamma) & \xrightarrow{\tau_\Gamma} & \mathbb{P}'(K(\Gamma)) \end{array}$$

The composition of BI hyperdoctrine morphisms $(K, \tau) : \mathbb{P} \to \mathbb{P}'$ and $(K', \tau') : \mathbb{P}' \to \mathbb{P}''$ is given by $(K' \circ K, \tau'_{K(-)} \circ \tau)$. This yields a category BIHyp. $\square$

For indexed (B)BI frames the definition splits into two because of the weakening of equality to a preorder on the intuitionistic side. It is straightforward to show that the notion of indexed BI frame morphism collapses to that for indexed BBI frames when the preorders $\preccurlyeq$ are substituted for $=$.

Definition 5.6 (Indexed BI Frame Morphism)**.** Given indexed BI frames $\mathcal{R} : \mathrm{C} \to \mathrm{BIFr}$ and $\mathcal{R}' : \mathrm{D} \to \mathrm{BIFr}$, an *indexed BI frame morphism* $(L, \lambda) : \mathcal{R} \to \mathcal{R}'$ is a pair (L, λ) such that:
(1) $L : D \to C$ is a finite product preserving functor;
(2) $\lambda : \mathcal{R} \circ L \to \mathcal{R}'$ is a natural transformation;
(3) (Lift Property) If there exists x and y such that $\mathcal{R}'(\Delta_X)(y) \preccurlyeq \lambda_{X\times X}(x)$ then there exists y' such that $\mathcal{R}(\Delta_{L(X)})(y') \preccurlyeq x$;
(4) (Morphism Pseudo Epi) If there exists x and y with $\mathcal{R}'(\pi_{\Gamma,X})(x) \preccurlyeq \lambda_\Gamma(y)$ then there exists z such that $x \preccurlyeq \lambda_{\Gamma\times X}(z)$ and $\mathcal{R}(\pi_{L(\Gamma),L(X)})(z) \preccurlyeq y$.

The composition of indexed BI frame morphisms $(L', \lambda') : \mathcal{R}' \to \mathcal{R}''$ and $(L, \lambda) : \mathcal{R} \to \mathcal{R}'$ is given by $(L \circ L', \lambda' \circ \lambda_{L'(-)})$. This yields a category IndBIFr. $\square$

Definition 5.7 (Indexed BBI Frame Morphism)**.** For indexed BBI frames $\mathcal{R} : \mathrm{C} \to \mathrm{BBIFr}$ and $\mathcal{R}' : \mathrm{D} \to \mathrm{BBIFr}$, an *indexed BBI frame morphism* $(L, \lambda) : \mathcal{R} \to \mathcal{R}'$ is a pair (L, λ) satisfying (1) and (2) of the previous definition as well as
(3′) (Lift Property′) if there exist x and y such that $\lambda_{X\times X}(x) = \mathcal{R}'(\Delta_X)(y)$, then there exists y' such that $\mathcal{R}((\Delta_{L(X)}))(y') = x$, and
(4′) (Quasi-Pullback) for all objects Γ and X in C, the following square is a quasi-pullback:

$$\begin{array}{ccc} \mathcal{R}(L(\Gamma) \times L(X)) & \xrightarrow{\lambda_{\Gamma\times X}} & \mathcal{R}(\Gamma \times X) \\ \downarrow{\scriptstyle \mathcal{R}(\pi_{L(\Gamma),L(X)})} & & \downarrow{\scriptstyle \mathcal{R}'(\pi_{\Gamma,X})} \\ \mathcal{R}(L(\Gamma)) & \xrightarrow{\lambda_\Gamma} & \mathcal{R}(\Gamma) \end{array}$$

The composition of indexed BBI frame morphisms $(L', \lambda') : \mathcal{R}' \to \mathcal{R}''$ and $(L, \lambda) : \mathcal{R} \to \mathcal{R}'$ is given by $(L \circ L', \lambda' \circ \lambda_{L'(-)})$. This yields a category Ind(B)BIFr. $\square$

We can now show that the ‘algebraic’ and ‘relational’ semantics of **FO(B)BI** generate a dual adjunction between these categories. First we define structures that correspond to the complex algebras and prime filter frames of Section 3, and show these transformations underpin functors. To obtain complex hyperdoctrines, we straightforwardly compose an indexed frame with the appropriate complex algebra functor from (B)BI.

Definition 5.8 (Complex BI Hyperdoctrine)**.** Given an indexed BI frame $\mathcal{R} : \mathrm{C} \to \mathrm{BIFr}$, the *complex hyperdoctrine* of $\mathcal{R}$, is given by $Com^{\mathrm{BI}}(\mathcal{R}(-)) : \mathrm{C}^{op} \to \mathrm{BIAlg}$, together with $Ran(\mathcal{R}(\Delta_X))$ as $=_X$, $\mathcal{R}(\pi_{\Gamma,X})^*$ as $\exists X_\Gamma$, and $\mathcal{R}(\pi_{\Gamma,X})_*$ as $\forall X_\Gamma$, where

$$\begin{aligned} \mathcal{R}(\pi_{\Gamma,X})^*(A) &= \{x \mid \text{there exists } y \in A : \mathcal{R}(\pi_{\Gamma,X})(y) \preccurlyeq x\} \text{ and} \\ \mathcal{R}(\pi_{\Gamma,X})_*(A) &= \{x \mid \text{for all } y,\ \text{if } x \preccurlyeq \mathcal{R}(\pi_{\Gamma,X})(y) \text{ then } y \in A\}. \end{aligned}$$

□

Given that the complex algebra operations thus far have matched the corresponding semantic clauses on frames, one might have expected $\exists X_\Gamma$ to be given by the direct image $\mathcal{R}(\pi_{\Gamma,X})$. Using the fact that $\mathcal{R}(\pi_{\Gamma,X})$ a BI morphism — specifically property (2) of Definition 3.5 — it can be shown that $\mathcal{R}(\pi_{\Gamma,X})^*$ is in fact identical to $\mathcal{R}(\pi_{\Gamma,X})$ so this is indeed the case. We use its presentation as $\mathcal{R}(\pi_{\Gamma,X})^*$ as it simplifies some proofs that follow.

Definition 5.9 (Complex BBI Hyperdoctrine)**.** Given an indexed BBI frame $\mathcal{R} : \mathrm{C} \to \mathrm{BBIFr}$, the *complex hyperdoctrine* of $\mathcal{R}$, is given by $Com^{\mathrm{BBI}}(\mathcal{R}(-)) : \mathrm{C}^{op} \to \mathrm{BBIAlg}$, together with $Ran(\mathcal{R}(\Delta_X))$ as $=_X$, $\mathcal{R}(\pi_{\Gamma,X})^*$ as $\exists X_\Gamma$, and $\mathcal{R}(\pi_{\Gamma,X})_*$ as $\forall X_\Gamma$, where

$$\begin{aligned} \mathcal{R}(\pi_{\Gamma,X})^*(A) &= \{x \mid \text{there exists } y \in A : \mathcal{R}(\pi_{\Gamma,X})(y) = x\} \text{ and} \\ \mathcal{R}(\pi_{\Gamma,X})_*(A) &= \{x \mid \text{for all } y,\ \text{if } x = \mathcal{R}(\pi_{\Gamma,X})(y) \text{ then } y \in A\}. \end{aligned}$$

□

Lemma 5.10. *Given an indexed (B)BI frame $\mathcal{R} : \mathrm{C} \to \mathrm{(B)BIFr}$, the complex hyperdoctrine $Com^{\mathrm{(B)BI}}(\mathcal{R}(-))$ is a (B)BI hyperdoctrine.*

Proof. We concentrate on the verifications relating to $\mathcal{R}(\pi_{\Gamma,X})^*$ and $\mathcal{R}(\pi_{\Gamma,X})_*$ of a BI complex hyperdoctrine. It is straightforward to see these map upwards-closed sets to upwards-closed sets and are monotone with respect to the subset ordering $\subseteq$. The adjointness properties follow from the definitions so it just remains to prove naturality.

We give the case for $\exists X_\Gamma$. Given a morphism $s : \Gamma \to \Gamma'$ in C and an element $A \in Com^{\mathrm{BI}}(\mathcal{R}(\Gamma' \times X))$, we must show $\mathcal{R}(\pi_{\Gamma,X})^*(\mathcal{R}(s \times id_X)^{-1}(A)) = \mathcal{R}(s)^{-1}(\mathcal{R}(\pi_{\Gamma',X})^*(A))$. Suppose $x \in \mathcal{R}^*(\pi_{\Gamma,X})(\mathcal{R}(s \times id_X)^{-1}(A))$: then there exists y such that $\mathcal{R}(\pi_{\Gamma,X})(y) \preccurlyeq x$ and $\mathcal{R}(s \times id_X)(y) \in A$. We have $\mathcal{R}(\pi_{\Gamma',X})(\mathcal{R}(s \times id_X)(y)) = \mathcal{R}(s)(\mathcal{R}(\pi_{\Gamma,X})(y)) \preccurlyeq \mathcal{R}(s)(x)$. Hence $x \in \mathcal{R}(s)^{-1}(\mathcal{R}(\pi_{\Gamma',X})^*(A))$, as required.

Conversely, assume $x \in \mathcal{R}(s)^{-1}(\mathcal{R}(\pi_{\Gamma',X})^*(A))$. Then there exists $y \in A$ such that $\mathcal{R}(\pi_{\Gamma',X})(y) \preccurlyeq \mathcal{R}(s)(x)$. Then by (Psuedo Epi), there exists z such that $\mathcal{R}(\pi_{\Gamma,X})(z) \preccurlyeq x$ and $y \preccurlyeq \mathcal{R}(s \times id_X)(z)$. By upwards-closure of A, $\mathcal{R}(s \times id_X)(z) \in A$. Hence we have $x \in \mathcal{R}(\pi_{\Gamma,X})^*(\mathcal{R}(s \times id_X)^{-1}(A))$, as required.

The proof for BBI complex hyperdoctrines follows immediately by substituting every instance of $\preccurlyeq$ with $=$ in the above argument, where the quasi pullback property allows us to assume the existence of z such that $\mathcal{R}(\pi_{\Gamma,X})(z) = x$ and $y = \mathcal{R}(s \times id_X)(z)$ from $\mathcal{R}(\pi_{\Gamma',X})(y) = \mathcal{R}(s)(x)$. □

Lemma 5.11. *Given a morphism of indexed (B)BI frames $(L, \lambda) : \mathcal{R} \to \mathcal{R}'$, $(L, \lambda^{-1}) : Com^{\mathrm{(B)BI}}(\mathcal{R}'(-)) \to Com^{\mathrm{(B)BI}}(\mathcal{R}(-))$ is a morphism of (B)BI hyperdoctrines.* □

Hence we obtain functors $\mathfrak{F} : \mathrm{(B)BIFr} \to \mathrm{(B)BIHyp}$ where $\mathfrak{F}(\mathcal{R})$ is the complex hyperdoctrine for objects $\mathcal{R}$ and $\mathfrak{F}(L, \lambda) = (L, \lambda^{-1})$ for morphisms (L, λ). In similar fashion, we can obtain indexed frames from hyperdoctrines by composing with the prime filter functors of (B)BI.

Definition 5.12 (Indexed Prime Filter (B)BI Frame)**.** Given a (B)BI hyperdoctrine $\mathbb{P}$, the *indexed prime filter frame* is given by $Pr^{\text{(B)BI}}(\mathbb{P}(-))$. □

Lemma 5.13. *Given a (B)BI hyperdoctrine* $\mathbb{P} : \mathrm{C}^{op} \to \mathrm{Poset}$ *the indexed prime filter frame* $Pr^{(B)BI}(\mathbb{P}(-))$ *is an indexed (B)BI frame.* □

Proof. We first show the (Pseudo Epi) property is satisfied when $\mathbb{P}$ is a BI hyperdoctrine. Assume we have objects Γ, Γ' and X in C and a morphism $s : \Gamma \to \Gamma'$. Let prime filters F_0 and F_1 be such that $\mathbb{P}(\pi_{\Gamma',X})^{-1}(F_1) \subseteq \mathbb{P}(s)^{-1}(F_0)$.

Consider the filter $f_2 = \{x \mid \exists a \in F_1, x \geq \mathbb{P}(s \times id_X)(a)\}$. Suppose for contradiction that f_2 is not proper. Then there exists $a \in F_1$ such that $\mathbb{P}(s \times id_X)(a) = \bot$. By adjointness, $\exists X_\Gamma(\bot) = \bot$, so $\mathbb{P}(s)(\exists X_{\Gamma'}(a)) = \exists X_\Gamma(\mathbb{P}(s \times id_X)(a)) = \bot$ by naturality. This entails $\exists X_{\Gamma'}(a) \notin \mathbb{P}(s)^{-1}(F_0)$ so $\exists X_{\Gamma'}(a) \notin \mathbb{P}(\pi_{\Gamma',X})^{-1}(F_1)$ by assumption. However, by adjointness and filterhood, $\mathbb{P}(\pi_{\Gamma',X})(\exists X_{\Gamma'}(a)) \in F_1$, a contradiction.

Clearly $\mathbb{P}(s \times id_X)^{-1}(f_2) \supseteq F_1$. To see that the other required inclusion holds, suppose $a \in \mathbb{P}(\pi_{\Gamma,X})^{-1}(f_2)$. Then there exists $b \in F_1$ such that $\mathbb{P}(s \times id_X)(b) \leq \mathbb{P}(\pi_{\Gamma,X})(a)$. By adjointness $\exists X_\Gamma(\mathbb{P}(s \times id_X)(b)) \leq a$ and so by naturality $\mathbb{P}(s)(\exists X_{\Gamma'}(b)) \leq a$. Since $\mathbb{P}(\pi_{\Gamma',X})(\exists X_{\Gamma'}(b)) \in F_1$, we have $\exists X_{\Gamma'}(b) \in \mathbb{P}(\pi_{\Gamma',X})^{-1}(F_1) \subseteq \mathbb{P}(s)^{-1}(F_0)$. Thus by filterhood, $a \in F_0$. Since f_2 is a filter, using Zorn's Lemma, it can be extended to a prime filter F_2 satisfying these properties.

For $\mathbb{P}$ a BBI hyperdoctrine, we instead start with the assumption of prime filters F_0 and F_1 such that $\mathbb{P}(\pi_{\Gamma',X})^{-1}(F_1) = \mathbb{P}(s)^{-1}(F_0)$. This is sufficient to once again prove the existence of a prime filter F_2 satisfying $\mathbb{P}(s \times id_X)^{-1}(F_2) \supseteq F_1$ and $\mathbb{P}(\pi_{\Gamma,X})^{-1}(F_2) \subseteq F_0$. However, maximality of prime filters on Boolean algebras collapses the inclusions to equalities — $\mathbb{P}(s \times id_X)^{-1}(F_2) = F_1$ and $\mathbb{P}(\pi_{\Gamma,X})^{-1}(F_2) = F_0$ — so the quasi pullback property holds. □

Lemma 5.14. *Given a morphism of (B)BI hyperdoctrines* $(K, \tau) : \mathbb{P} \to \mathbb{P}'$*,* $(K, \tau^{-1}) : Pr(\mathbb{P}'(-)) \to Pr(\mathbb{P}(-))$ *is a morphism of indexed (B)BI frames.* □

This gives us the corresponding functors $\mathfrak{G} : \text{(B)BIHyp} \to \text{(B)BIFr}$, where $\mathfrak{G}(\mathbb{P})$ is the indexed prime filter frame $Pr^{\text{(B)BI}}(\mathbb{P}(-))$ for objects $\mathbb{P}$ and $\mathfrak{G}(K, \tau) = (K, \tau^{-1})$ for morphisms (K, τ).

We can finally define natural transformations $\Theta : Id_{\text{(B)BIHyp}} \to \mathfrak{F}\mathfrak{G}$ and $\mathrm{H} : Id_{\text{Ind(B)BIFr}} \to \mathfrak{G}\mathfrak{F}$ straightforwardly from those given for **(B)BI**: $\Theta_{\mathbb{P}} = (Id_{\mathrm{C}}, \theta_{\mathbb{P}(-)})$ and $\mathrm{H}_{\mathcal{R}} = (Id_{\mathrm{C}}, \eta_{\mathcal{R}(-)})$. The property of being a natural transformation is inherited from θ and η, so the last non-trivial verification before stating the dual adjunction theorem is that these define morphisms of the right sort at each $\mathbb{P}$ and $\mathcal{R}$.

Lemma 5.15.

(1) $\Theta_{\mathbb{P}} = (Id_{\mathrm{C}}, \theta_{\mathbb{P}(-)})$ *is a morphism of (B)BI hyperdoctrines.*
(2) $\mathrm{H}_{\mathcal{R}} = (Id_{\mathrm{C}}, \eta_{\mathcal{R}(-)})$ *is a morphism of indexed (B)BI frames.* □

The dual adjunction for (B)BI algebras and frames then entails the dual adjunction for (B)BI hyperdoctrines and indexed (B)BI frames.

Theorem 5.16. *The functors* $\mathfrak{F}$ *and* $\mathfrak{G}$ *and the natural transformations* Θ *and* H *form a dual adjunction of categories between* (B)BIHyp *and* Ind(B)BIFr*.*

We now make the connection to the respective semantics clear. Given any interpretation on an indexed BI frame $[\![-]\!]$, we automatically have a interpretation for the complex hyperdoctrine as predicate symbols are interpreted as upwards-closed subsets; that is, elements of complex algebras of BI frames. The same is true for indexed BBI frames as the interpretation of predicate symbols are subsets and thus elements of complex algebras of BBI frames. A simple inductive argument shows that satisfaction coincides for these models.

Proposition 5.17. *Given an indexed (B)BI frame $\mathcal{R}$ and an interpretation $[\![-]\!]$, for all **FO(B)BI** formulas ϕ in context Γ and $x \in \mathcal{R}([\![\Gamma]\!])$, $x, [\![-]\!] \vDash^{\Gamma} \phi$ iff $x \in [\![\phi]\!]$.* □

Conversely, given an interpretation $[\![-]\!]$ on a (B)BI hyperdoctrine, an interpretation $\widetilde{[\![-]\!]}$ can be defined by setting $\widetilde{[\![P]\!]} = \theta_{\mathbb{P}([\![X_1]\!]\times\cdots\times[\![X_m]\!])}([\![P]\!])$ for each predicate symbol of type $X_1, \ldots, X_m$. The following proposition is a corollary of the dual adjunction.

Proposition 5.18. *Given a (B)BI hyperdoctrine $\mathbb{P}$ and an interpretation $[\![-]\!]$, for all **FO(B)BI** formulas ϕ in context Γ and prime filters F of $\mathbb{P}([\![\Gamma]\!])$, $[\![\phi]\!] \in F$ iff $F, \widetilde{[\![-]\!]} \vDash^{\Gamma} \phi$.* □

Corollary 5.19 (Relational Soundness and Completeness)**.** *For all **FO(B)BI** formulas ϕ, ψ in context Γ, $\phi \vdash^{\Gamma} \psi$ is provable in* $\mathrm{FO(B)BI_H}$ *iff $\phi \vDash^{\Gamma} \psi$.*

To obtain a dual equivalence of categories, we must move from indexed (B)BI frames to indexed (B)BI *spaces*.

Definition 5.20 (Indexed (B)BI Space)**.** An *indexed (B)BI space* is a functor $\mathcal{R} : \mathrm{C} \to$ (B)BISp such that

(1) $U \circ \mathcal{R} : \mathrm{C} \to$ (B)BIFr is an indexed (B)BI frame, where U : (B)BISp $\to$ (B)BIFr is the functor that forgets topological structure.
(2) For each object X in C, $Ran(\mathcal{R}(\Delta_X))$ is clopen;
(3) For each pair of objects Γ and X in C, $\mathcal{R}(\pi_{\Gamma,X})^*$ and $\mathcal{R}(\pi_{\Gamma,X})_*$ map
   a) (for indexed BI spaces) upwards-closed clopen sets to upwards-closed clopen sets.
   b) (for indexed BBI spaces) clopen sets to clopen sets. □

Indexed (B)BI space morphisms are given identically to indexed (B)BI frame morphisms (relativized to the category (B)BISp) and this yields a category Ind(B)BISp. By uniformly substituting the functors and natural transformations of the (B)BI dual adjunction with those of the (B)BI duality in the definitions of the (B)BI hyperdoctrine dual adjunction, we obtain the maps required for the duality theorem.

Given an indexed (B)BI space $\mathcal{R}$, $\mathfrak{F}(\mathcal{R})$ is the (B)BI hyperdoctrine given by the functor $Clop^{\mathrm{(B)BI}}(\mathcal{R}(-))$ together with $Ran(\mathcal{R}(\Delta_X))$ as $=_X$, $\mathcal{R}(\pi_{\Gamma,X})^*$ as $\exists X_\Gamma$ and $\mathcal{R}(\pi_{\Gamma,X})_*$ as $\forall X_\Gamma$. That this is well defined is immediate from the coherence conditions (2) and (3) in the definition of indexed (B)BI space, together with Lemma 5.10. $\mathfrak{F}(L, \lambda) = (L, \lambda^{-1})$ as before.

Given a (B)BI hyperdoctrine $\mathbb{P}$, $\mathfrak{G}(\mathbb{P}) = PrSp^{\mathrm{(B)BI}}(\mathbb{P}(-))$. That this satisfies properties (2) and (3) in the definition of indexed space can be derived from the fact that — because of Stone (Esakia) duality — the (upwards-closed) clopen sets of the prime filter space of $\mathbb{A}$ are all of the form $\theta_{\mathbb{A}}(a)$ for $a \in A$. It is then straightforward to use the adjointness properties of a (B)BI hyperdoctrine to show that $Ran(\mathbb{P}(\Delta_X)^{-1}) = \theta_{\mathbb{P}(X\times X)}(=_X)$, $(\mathbb{P}(\pi_{\Gamma,X})^{-1})^*(h_{\mathbb{P}(\Gamma\times X)}(a)) = \theta_{\mathbb{P}(\Gamma)}(\exists X_\Gamma(a))$ and $(\mathbb{P}(\pi_{\Gamma,X})^{-1})_*(h_{\mathbb{P}(\Gamma\times X)}(a)) = \theta_{\mathbb{P}(\Gamma)}(\forall X_\Gamma(a))$. $\mathfrak{G}(K, \tau) = (K, \tau^{-1})$ once again.

Finally, Θ and H are defined as in the dual adjunction except that θ and η are taken from **(B)BI** duality instead. Since these new Θ and H inherit the property of being natural isomorphisms from θ and η, we obtain **FO(B)BI** duality.

Theorem 5.21 (Duality Theorem for **FO(B)BI**)**.** *The categories of (B)BI hyperdoctrines and indexed (B)BI spaces are dually equivalent.* □

Given that the structures we have defined comprise the most general classes of bunched logic model, to what extent can we restrict the dual adjunctions and dualities to a subclass of model that correspond specifically to the standard model of Separation Logic? There are a number of properties that the (classical) memory model satisfies implicitly, given as follows by Brotherston & Villard [13]:

Partial deterministic:	$w, w' \in w_1 \circ w_2$ implies $w = w'$
Cancellativity:	$w \circ w_1 \cap w \circ w_2$ implies $w_1 = w_2$
Indivisible Units:	$(w \circ w') \cap E \neq \emptyset$
Disjointness:	$w \circ w \neq \emptyset$ implies $w \in E$
Divisibility:	for every $w \notin E$, there are $w_1, w_2 \notin E$ such that $w \in w_1 \circ w_2$
Cross Split:	whenever $(t \circ u) \cap (v \circ w) \neq \emptyset$, there exist tv, tw, uv, uw such that $t \in tv \circ tw, u \in uv \circ uw, v \in tv \circ uv$ and $w \in tw \circ uw$.

Here we do not consider their property Single Unit as it is only satisfied by the propositional heap model, and not by the predicate store-heap models. Brotherston & Villard show that while Divisibility and Indivisible Units are definable in **BBI**, Partial Deterministic, Cancellativity and Disjointness are not, with cross split conjectured to be similarly undefinable.

It is straightforward to restrict the dual adjunction and duality theorems to the algebras satisfying axioms corresponding to Indivisible Units ($\mathrm{I} \wedge (a * b) \leq a$) and Divisibility ($\neg\mathrm{I} \leq \neg\mathrm{I} * \neg\mathrm{I}$). However, in the other cases the undefinability results preclude this possibility: no algebraic axiom can possibly pick out these classes of model.

The extent to which the remaining properties define different notions of validity has been partially investigated by Larchey-Wendling & Galmiche [54]. In particular, they show that the formula $\mathcal{I} * \mathcal{I} \to \mathcal{I}$, where $\mathcal{I} = \neg(\top \twoheadrightarrow \neg\mathrm{I})$, distinguishes partial determinstic models from non-deterministic models. In summary, this situation isn't totally benign.

Brotherston & Villard's solution is to give a conservative extension of **BBI** in the spirit of hybrid logic [3] called **HyBBI**. The additional expressivity of nominals and satisfaction operators allows the logic to pick out specific states of the model, making the axiomatization of the remaining properties possible. Thus, in order to precisely capture the concrete model of Separation Logic we would have to extend the techniques of the present work to hybrid extensions of bunched logics. This would also enable us to connect our work to the extensive tableaux proof theory given for bunched logics [40, 53, 27, 38] via the close connection between hybrid extensions and labelled proof systems. Although duality theorems for axiomatic extensions of hybrid logic with one unary modality have been given [23], the generalization required to achieve such a result for hybrid bunched logics is beyond the scope of this paper. We defer such an investigation to another occasion.

6. Multiplicative Extensions

To conclude, we adapt the results of Section 4 to a variety of logics that extend **BI** and **BBI**. Applications of these logics include reasoning about deny-guarantee permissions, concurrency and — via the interpretation of heap intersection operations — complex resource sharing. We concentrate on three: the *classical* bunched logics that extend **(B)BI** with a De Morgan negation, facilitating the definition of multiplicatives corresponding to disjunction and falsum [11]; the family of *sub-classical* bunched logics that sit intermediate between **(B)BI** and the classical bunched logics; and finally **CKBI**, a new logic suggested by algebraic interpretations of (a basic version of) Concurrent Separation Logic [59].

For reasons of space we only sketch arguments and do not give full details of the topological structures that allow us to restrict the dual adjunctions to dual equivalences of categories: it is, however, straightforward to do so by a careful choice of topological coherence conditions for the additional frame operations. We leave it to the reader to fill in these gaps.

6.1. **Classical Bunched Logics.** Brotherston & Calcagno [11] introduces the logic **CBI** that extends **BBI** with a De Morgan negation $\sim$. By substituting $*$ and I into the De Morgan laws relating $\wedge$ to $\vee$ and $\top$ to $\bot$, this yields multiplicative connectives corresponding to disjunction and falsum. Although it is shown that the Separation Logic heap model is not a model of **CBI**, a number of interesting applications are suggested ranging from deny-guarantee permissions to regular languages. Brotherston [10] also gives a display calculus for **DMBI**, the apparent intuitionistic variant of **CBI** that instead extends **BI**, but no Kripke semantics or completeness proof can be found in the literature.

We give the syntax and Hilbert-type proof theory of the logics by defining the algebras they determine.

Definition 6.1 (DMBI/CBI Algebra)**.** A *DMBI (CBI) algebra* is a (B)BI algebra $\mathbb{A}$ extended with an operation $\sim: A \to A$ satisfying, for all $a \in A$, $\sim a = a \multimap\!\!\bullet \sim \mathrm{I}$ and $\sim\sim a = a$. □

On such an algebra we can define a multiplicative falsum $\bot^* = \sim \mathrm{I}$ and a multiplicative disjunction $a \mathbin{\rotatebox[origin=c]{180}{$\&$}} b = \sim(\sim a \bullet \sim b)$. $\sim$ also makes $(A, \wedge, \vee, \sim, \top, \bot)$ a De Morgan algebra: the fact that $\multimap\!\!\bullet$ converts joins in its left argument into meets entails

$$\sim(a \vee b) = (a \vee b) \multimap\!\!\bullet \bot^* = (a \multimap\!\!\bullet \bot^*) \wedge (b \multimap\!\!\bullet \bot^*) = \sim a \wedge \sim b\,.$$

It then follows that $a \leq b$ iff $\sim b \leq \sim a$, and thus also $\sim(a \wedge b) = \sim a \vee \sim b$. Hence $\sim$ is a dual automorphism on the underlying bounded distributive lattice of $\mathbb{A}$. It is also straightforward to derive the following property from the residuation property of $\multimap\!\!\bullet$ and the equation $\sim a = a \multimap\!\!\bullet \bot^*$.

Proposition 6.2. *Given a DMBI (CBI) algebra $\mathbb{A}$, for all $a, b, c \in A$,*

$$a \bullet b \leq c \textit{ iff } b \bullet \sim c \leq \sim a \textit{ iff } a \bullet \sim c \leq \sim b.$$

□

Definition 6.3 (DMBI/CBI Frame)**.** A *DMBI frame* is a BI frame extended with an operation $n : X \to X$ satisfying

$$\text{(Dual)}\quad x \preccurlyeq y \to n(y) \preccurlyeq n(x) \qquad\qquad \text{(Invol)}\quad n(n(x)) = x$$
$$\text{(Compat)}\quad z \in x \circ y \leftrightarrow n(x) \in y \circ n(z) \leftrightarrow n(y) \in x \circ n(z).$$

A *CBI frame* is a BBI frame extended with an operation $n : X \to X$ satisfying (Invol) and (Compat). □

The definition of CBI frame here looks different to the notion given by Brotherston & Calcagno [11] but is equivalent. There, a (multi-unit) CBI model is a tuple $(X, \circ, E, n, \infty)$ such that $(X, \circ, E)$ is a BBI frame, with $n : X \to X$ and $\infty \subseteq X$ satisfying, for all $x \in X$, $n(x)$ is the unique element such that $\infty \cap (n(x) \circ x) \neq \emptyset$. (Invol) and (Compat) are then proved as consequences of this definition in Proposition 2.3 (1) and (3). As they discuss, the choice of ∞ is fixed by the choice of n, and it can easily be seen that defining $\infty = \{n(e) \mid e \in E\}$ on our CBI frames yields their CBI models. We choose our presentation as it simplifies proofs.

The semantics for **DMBI** (**CBI**) is given by extending the semantics on (B)BI frames for **(B)BI** by the clause

$$x \vDash \sim\phi \text{ iff } n(x) \nvDash \phi.$$

Semantic clauses can then straightforwardly be given for the derived operations $\stackrel{*}{\vee}$ and $\perp^*$ using the clauses for $\sim, *$ and I. In the case of **DMBI**, persistence for formulas of the form $\sim\phi$ is guaranteed by the frame property (Dual), and thus also for formulas of the form $\phi \stackrel{*}{\vee} \psi$ and $\perp^*$. It is then simple to define complex algebras for DMBI and CBI frames.

Definition 6.4 (DMBI/CBI Complex Algebra)**.** Given a DMBI (CBI) frame $\mathcal{X}$, the complex algebra $Com^{\text{DMBI}}(\mathcal{X})$ ($Com^{\text{CBI}}(X)$) is given by extending $Com^{\text{(B)BI}}(\mathcal{X})$ with the operation $\sim_{\mathcal{X}}$, defined $\sim_{\mathcal{X}}(A) = \{x \mid n(x) \notin A\}$. $\square$

Lemma 6.5. *Given a DMBI (CBI) frame $\mathcal{X}$, $Com^{\text{DMBI}}(\mathcal{X})$ ($Com^{\text{CBI}}(X)$) is a DMBI (CBI) algebra.*

Proof. We give the case for **DMBI**: the **CBI** case follows as a corollary. First note that (Dual) ensures $\sim_{\mathcal{X}}$ maps upwards-closed sets to upwards-closed sets and thus $Com^{DMBI}(\mathcal{X})$. That $\sim_{\mathcal{X}}\sim_{\mathcal{X}}(A) = A$ is trivial so we finish by verifying $\sim_{\mathcal{X}}(A) = A \multimapdot_{\mathcal{X}} \sim_{\mathcal{X}} E$. First assume $a \in \sim_{\mathcal{X}}(A)$ and let $a \preccurlyeq a'$ with $c \in a' \circ b$ and $b \in A$: we need to show $n(c) \notin E$ so that $a \in A \multimapdot_{\mathcal{X}} \sim_{\mathcal{X}} E$. Suppose for contradiction $n(c) \in E$. By (Compat) we have $n(a') \in b \circ n(c)$ and by the BI frame property (Contr) this entails $b \preccurlyeq n(a')$. By (Invol), $n(a') \preccurlyeq n(a)$ so by upwards closure of A, $n(a) \in A$, a contradiction. Hence $n(c) \notin E$.

In the other direction, assume $a \in A \multimapdot_{\mathcal{X}} \sim_{\mathcal{X}} E$. By the BI frame property (Weak), there exists $e \in E$ such that $a \in a \circ e$. By (Compat), $n(e) \in a \circ n(a)$. Hence if $n(a) \in A$, we would have $n(e) \in \sim_{\mathcal{X}}(E) = \{x \mid n(x) \notin E\}$, a contradiction since $n(n(e)) = e$. Hence $n(a) \notin A$ so $a \in \sim(A)$. $\square$

We can also use $\sim$ to define an $n_{\mathbb{A}}$ for the prime filter frame of a DMBI (CBI) algebra. For a set $X \subseteq A$, let $\sim X = \{\sim x \mid x \in X\}$ and $\overline{X} = \{x \mid x \notin X\}$.

Definition 6.6 (Prime Filter DMBI/CBI Frame)**.** Given a DMBI (CBI) algebra $\mathbb{A}$, the prime filter frame $Pr^{\text{DMBI}}(\mathbb{A})$ ($Pr^{\text{CBI}}(\mathbb{A})$) is given by extending the prime filter frame $Pr^{(B)BI}(\mathbb{A})$ with the operation $n_{\mathbb{A}}$, defined $n_{\mathbb{A}}(F) = \overline{\sim F}$.

That this is well defined follows from the fact $\sim$ is a dual automorphism on the underlying bounded distributive lattice: this entails that, given a prime filter F, $\sim F$ is a prime ideal, and thus $\overline{\sim F}$ is a prime filter.

Lemma 6.7. *Given a DMBI (CBI) algebra $\mathbb{A}$, $Pr^{\text{DMBI}}(\mathbb{A})$ ($Pr^{\text{CBI}}(\mathbb{A})$) is a DMBI (CBI) frame.*

Proof. Once again we restrict to the case for **DMBI** as the **CBI** case follows as a corollary. It is trivial that $n_{\mathbb{A}}$ satisfies (Dual) and (Invol). We give a characteristic portion of (Compat):

$n_{\mathbb{A}}(F_0) \in F_1 \circ_{\mathbb{A}} n_{\mathbb{A}}(F_2)$ iff $n_{\mathbb{A}}(F_1) \in F_0 \circ_{\mathbb{A}} n_{\mathbb{A}}(F_2)$. First assume the left hand side. Then for all $a_1 \in F_1$ and $a_2 \not\in \sim F_2$, $\sim(a_1 \bullet a_2) \not\in F_0$, since $a_1 \bullet a_2 \in n_{\mathbb{A}}(F_0)$ by assumption and $\sim\sim(a_1 \bullet a_2) = a_1 \bullet a_2$. This entails, for any $a_0 \in F_0$, that $a_0 \not\leq \sim(a_1 \bullet a_2)$, and (since $\sim$ is a dual automorphism on the underlying lattice), $a_1 \bullet a_2 \not\leq \sim a_0$. By Proposition 6.2 it follows that, for all $a_0 \in F_0$, $\sim a_1 \in \sim F_1$ and $a_2 \in n_{\mathbb{A}}(F_2)$, $a_0 \bullet a_2 \not\leq \sim a_1$. Thus, $a_0 \bullet a_2 \in \sim F_1$ would imply $a_0 \bullet a_2 \not\leq \sim\sim(a_0 \bullet a_2) = a_0 \bullet a_2$ so $a_0 \bullet a_2 \not\in \sim F_1$ and thus $a_0 \bullet a_2 \in n_{\mathbb{A}}(F_1)$, as required.

Next, assuming the right hand side, we can similarly reason that for $\sim a_0 \in \sim F_0, a_1 \in F_1$ and $a_2 \in n_{\mathbb{A}}(F_2)$, $a_1 \bullet a_2 \not\leq \sim a_0$. Hence $a_1 \bullet a_2 \not\in \sim F_0$ and so $a_1 \bullet a_2 \in n_{\mathbb{A}}(F_0)$ as required. □

Extending the notion of (B)BI morphism to DMBI (CBI) is straightforward: we just require the DMBI (CBI) morphisms to additionally respect n: $g(n(x)) = n'(g(x))$. Then we can adapt the functors and natural transformations of the (B)BI dual adunction to this case: the functor action on morphisms gives the inverse image maps, and θ and η are defined in the same way.

Theorem 6.8 (Soundness and Completeness for **DMBI** and **CBI**)**.** *The functors Pr^{DMBI} (Pr^{CBI}), Com^{DMBI} (Com^{CBI}) and natural transformations θ and η form a dual adjunction of categories between* DMBIAlg *(*CBIAlg*) and* DMBIFr *(*CBIFr*).* □

Duality follows by taking the obvious extension of (B)BI space with the DMBI (CBI) frame structure, satisfying the coherence condition that n maps clopen sets to clopen sets.

6.2. **Sub-Classical Bunched Logics.** Brotherston & Villard [14] introduces a family of logics extending **BBI** that they call *sub-classical bunched logics*, as they lie intermediate between **BBI** and **CBI**. As heaps equipped with intersection operations are models of the logics, they are of clear interest to the Separation Logic community, with verification of algorithms involving complex resource sharing suggested as a natural application. Basic Bi-intuitionistic Boolean Bunched logic (**BiBBI**) is defined to be the multiplicative extension of **BBI** that drops all De Morgan laws between multiplicative conjunction, disjunction, truth and falsum. Thus in **BiBBI** these connectives can no longer be defined in terms of each other (as they were for **DMBI** and **CBI**) and must be given as primitives. A number of these correspondences can then be added as axioms without the logic collapsing into **CBI**. We show that our framework captures **BiBBI** and its axiomatic extensions, as well as the evident intuitionistic variant **BiBI** and the intermediate logics weaker than **DMBI**.

Definition 6.9 (Basic Bi(B)BI Algebra)**.** A *basic Bi(B)BI algebra* $\mathbb{A}$ is a (B)BI algebra extended with operations $\mathbin{\vee\!\!\!*}$, $\mathbin{\setminus^{*}}$, and a constant $\perp^*$, such that $\mathbin{\vee\!\!\!*}$ is commutative and, for all $a, b, c \in A$, $a \leq b \mathbin{\vee\!\!\!*} c$ iff $a \mathbin{\setminus^{*}} b \leq c$. □

Residuation of $\mathbin{\vee\!\!\!*}$ and $\mathbin{\setminus^{*}}$ guarantees that $\mathbin{\vee\!\!\!*}$ is order-preserving in each argument, as well as a number of other properties dual to those of Proposition 3.2.

Proposition 6.10. *Let $\mathbb{A}$ be a basic Bi(B)BI algebra. Then, for all $a, b, a', b' \in A$ and $X, Y \subseteq A$, we have the following:*

(1) *If $a \leq a'$ and $b \leq b'$ then $a \mathbin{\vee\!\!\!*} b \leq a' \mathbin{\vee\!\!\!*} b'$;*
(2) *If $\bigwedge X$ and $\bigwedge Y$ exist then $\bigwedge_{x \in X, y \in Y} x \mathbin{\vee\!\!\!*} y$ exists and $(\bigwedge X) \mathbin{\vee\!\!\!*} (\bigwedge Y) = \bigwedge_{x \in X, y \in Y} x \mathbin{\vee\!\!\!*} y$;*
(3) *If $a = \top$ or $b = \top$ then $a \mathbin{\vee\!\!\!*} b = \top$;*
(4) *If $\bigwedge X$ exists then for any $z \in A$: $\bigvee_{x \in X}(x \mathbin{\setminus^{*}} z)$ exists with $\bigvee_{x \in X}(x \mathbin{\setminus^{*}} z) = (\bigwedge X) \mathbin{\setminus^{*}} z$;*
(5) *If $\bigvee X$ exists then for any $z \in A$ $\bigvee_{x \in X}(z \mathbin{\setminus^{*}} x)$ exists with $\bigvee_{x \in X}(z \mathbin{\setminus^{*}} x) = z \mathbin{\setminus^{*}} (\bigvee X)$; and*
(6) *$a \mathbin{\setminus^{*}} \top = \perp \mathbin{\setminus^{*}} a = \perp$.* □

$$
\begin{array}{rcll}
r & \vDash & \bot^* & \text{iff } r \notin U \\
r & \vDash & \phi \,\text{\textasteriskcentered}\!\!\vee\, \psi & \text{iff for all } s,t,u,\ r \preccurlyeq s \in t \triangledown u \text{ implies } t \vDash \phi \text{ or } u \vDash \psi \\
r & \vDash & \phi \,\overset{*}{\setminus}\, \psi & \text{iff there exist } s,t,u \text{ such that } r \succcurlyeq s,\ u \in t \triangledown s,\ u \vDash \phi \text{ and } t \nvDash \psi
\end{array}
$$

Figure 8: Satisfaction for **Bi(B)BI**. **BiBBI** is the case where $\preccurlyeq$ is $=$.

Property	Axiom	Frame Correspondent
Associativity	$a \overset{*}{\vee} (b \overset{*}{\vee} c) \leq (a \overset{*}{\vee} b) \overset{*}{\vee} c$	$t' \preccurlyeq t \in x \triangledown y \wedge w \in t' \triangledown z \rightarrow$
		$\exists s, s', w'(s' \preccurlyeq s \in y \triangledown z \wedge w \preccurlyeq w' \in x \triangledown s')$
Unit Weakening	$a \leq a \overset{*}{\vee} \bot^*$	$u \in U \wedge x \in y \triangledown u \rightarrow x \preccurlyeq y$
Unit Contraction	$a \overset{*}{\vee} \bot^* \leq a$	$\exists u \in U (w \in w \triangledown u)$
Contraction	$a \overset{*}{\vee} a \leq a$	$x \in x \triangledown x$
Weak Distributivity	$a \bullet (b \overset{*}{\vee} c) \leq (a \bullet b) \overset{*}{\vee} c$	$t' \succcurlyeq t \in x_1 \circ x_2 \wedge t' \preccurlyeq t'' \in y_1 \triangledown y_2 \rightarrow$
		$\exists w (y_1 \in x_1 \circ w \wedge x_2 \in w \triangledown y_2)$

Figure 9: **Bi(B)BI** properties and axioms (cf. [14]). The **BiBBI** variants replace $\preccurlyeq$ with $=$.

Definition 6.11 (Basic Bi(B)BI Frame)**.** A *basic BiBI frame* is a given by a tuple $\mathcal{X} = (X, \preccurlyeq, \circ, E, \triangledown, U)$ such that $(X, \preccurlyeq, \circ, E)$ is a BI frame, $\triangledown : X^2 \rightarrow \mathcal{P}(X)$ and $U \subseteq X$, satisfying

$$\text{(Comm)} \quad z \in x \triangledown y \rightarrow z \in y \triangledown x; \quad \text{(Up)} \quad u \notin U \wedge u \preccurlyeq u' \rightarrow u' \notin U;$$

A *basic BiBBI frame* (cf. [14]) is a tuple $\mathcal{X} = (X, \circ, E, \triangledown, U)$, such that $(X, \circ, E)$ is a BBI frame, $\triangledown : X^2 \rightarrow \mathcal{P}(X)$ satisfies (Comm) and $U \subseteq X$. □

The semantics for **Bi(B)BI** extends that of **(B)BI**: the additional semantic clauses are given in Fig 8. It can be shown that for **BiBI** persistence extends to all formulas, as one would expect. **Bi(B)BI** is extended with axioms corresponding to properties of Bi(B)BI frames. These are given in Fig 9, with the axioms presented as inequalities on Bi(B)BI algebras.

Let Σ denote any combination of properties from Figure 9. By Bi(B)BIAlg_Σ we denote the category of basic Bi(B)BI algebras satisfying the axioms in Σ; likewise, by Bi(B)BIFr_Σ we denote the category of Bi(B)BI frames satisfying the corresponding frame properties in Σ, where Bi(B)BI morphisms are defined by appropriately extending (B)BI morphisms. We will show that for any Σ, a dual adjunction exists between Bi(B)BIAlg_Σ and Bi(B)BIFr_Σ; that is, $\textbf{Bi(B)BI} + \Sigma$ is sound and complete for its Kripke semantics.

Definition 6.12 (Bi(B)BI Complex Algebra)**.** Given a Bi(B)BI frame $\mathcal{X}$ the complex algebra of $\mathcal{X}$, $Com^{\text{Bi(B)BI}}(\mathcal{X})$, is given by extending the $Com^{\text{(B)BI}}(\mathcal{X})$ with the following:
(a) For BiBI: $\overset{*}{\vee}_\mathcal{X}, \overset{*}{\setminus}_\mathcal{X}, \bot^*_\mathcal{X}$, defined as

$$
\begin{array}{rl}
A \overset{*}{\vee}_\mathcal{X} B & = \{x \mid \text{ for all } s,t,u, x \preccurlyeq s \in t \triangledown u \text{ implies } t \in A \text{ or } u \in B\} \\
A \overset{*}{\setminus}_\mathcal{X} B & = \{x \mid \text{there exists } s,t,u \text{ s.t. } x \succcurlyeq s, u \in t \triangledown s, u \in A \text{ and } t \notin B\} \\
\bot^*_\mathcal{X} & = \{x \mid x \notin U\};
\end{array}
$$

(b) For BiBBI: $\overset{*}{\vee}_\mathcal{X}, \overset{*}{\setminus}_\mathcal{X}, \bot^*_\mathcal{X}$ as above, with $\preccurlyeq$ substituted for $=$ throughout the definitions. □

Lemma 6.13.

(1) *Given a basic Bi(B)BI frame $\mathcal{X}$, $Com^{\mathrm{Bi(B)BI}}(\mathcal{X})$ is a basic Bi(B)BI algebra.*

(2) *If $\mathcal{X}$ satisfies any frame property of Figure 9, $Com^{\mathrm{Bi(B)BI}}(\mathcal{X})$ satisifies the corresponding axiom.*

Proof. (1) is straightforward. For (2) we focus on the case of weak distributivity for BiBI frames, which collapses to the BiBBI variant when $\preccurlyeq$ is $=$. Let $t' \in A \bullet_{\mathcal{X}} (B \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}}_{\mathcal{X}} C)$. Then $t' \succcurlyeq t \in x_1 \circ x_2$ for some $x_1 \in A$ and $x_2 \in B \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}}_{\mathcal{X}} C$. Suppose $t' \preccurlyeq t'' \in y_1 \triangledown y_2$. We must show $y_1 \in A \bullet_{\mathcal{X}} B$ or $y_2 \in C$. Suppose $y_2 \notin C$. By the weak distributivity frame property, there exists w such that $y_1 \in x_1 \circ w$ and $x_2 \in w \triangledown y_2$. Since $y_2 \notin C$ and $x_2 \in B \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}}_{\mathcal{X}} C$ it follows that $w \in B$. Hence $y_1 \in A \bullet_{\mathcal{X}} B$ as required, and so $t' \in (A \bullet_{\mathcal{X}} B) \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}}_{\mathcal{X}} C$. □

Definition 6.14 (Prime Filter Bi(B)BI Frame)**.** Given a basic Bi(B)BI algebra $\mathbb{A}$, the prime filter frame of $\mathbb{A}$, $Pr^{\mathrm{Bi(B)BI}}(\mathbb{A})$, is given by extending $Pr^{\mathrm{(B)BI}}(\mathbb{A})$ with the operation $\triangledown_{\mathbb{A}}$, defined

$$F \triangledown_{\mathbb{A}} F' = \{F'' \mid \forall a, b \in A : a \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} b \in F'' \text{ implies } a \in F \text{ or } b \in F'\}$$

and the set $U_{\mathbb{A}} = \{F \mid \perp^* \notin F\}$. □

Lemma 6.15.

(1) *Given a basic Bi(B)BI algebra $\mathbb{A}$, $Pr^{Bi(B)BI}(\mathbb{A})$ is a basic Bi(B)BI frame.*

(2) *If $\mathbb{A}$ satisfies any axiom of Figure 9, $Pr^{Bi(B)BI}(\mathbb{A})$ satisfies the corresponding frame property.*

Proof. Once again we restrict ourselves to the non-trivial (2). We focus once more on the weak distributivity property for **BiBI**. Suppose $F_{t'} \supseteq F_t \in F_{x_1} \circ_{\mathbb{A}} F_{x_2}$ and $F_{t'} \subseteq F_{t''} \in F_{y_1} \triangledown_{\mathbb{A}} F_{y_2}$. Consider the set $f_w = \{b \mid \exists d \notin F_{y_2} (b \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} d \in F_{x_2})\}$.

Note f_w is a proper filter. First, it is upwards-closed since $\mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}}$ is order preserving. To see that it is closed under meets, suppose $b, b' \in f_w$. Then there exist $d, d' \notin F_{y_2}$ such that $b \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} d, b' \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} d' \in F_{x_2}$. WOLOG we may assume $d = d'$: F_{y_2} is prime so $d \vee d' \notin F_{y_2}$, and since $\mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}}$ is order preserving we have $b \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} (d \vee d'), b' \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} (d \vee d') \in F_{x_2}$. Now by Proposition 6.9 $(b \wedge b') \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} d = (b \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} d) \wedge (b' \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} d) \in F_{x_2}$, so $b \wedge b' \in f_w$. Finally, suppose for contradiction that $\perp \in f_w$. Then there exists $d \notin F_{x_2}$ such that $\perp \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} d \in F_{x_2}$. Letting $a \in F_{x_1}$ be arbitrary, by weak distributivity we have $a \bullet (\perp \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} d) \leq (a \bullet \perp) \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} d = \perp \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} d \in F_t \subseteq F_{t'}$. Thus $\perp \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} d \in F_{t'} \subseteq F_{t''}$ but $\perp \notin F_{y_1}$ and $d \notin F_{y_2}$, contradicting that $F_{t''} \in F_{y_1} \triangledown_{\mathbb{A}} F_{y_2}$.

Now $F_{y_1} \in F_{x_1} \circ_{\mathbb{A}} f_w$: let $a \in F_{x_1}$ and $b \in f_w$. Then there exists $d \notin F_{y_2}$ such that $b \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} d \in F_{x_2}$. Hence $a \bullet (b \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} d) \leq (a \bullet b) \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} d \in F_t \subseteq F_{t'} \subseteq F_{t''}$, and since $d \notin F_{y_2}$ it must be that $a \bullet b \in F_{y_1}$. We also have $F_{x_2} \in f_w \triangledown_{\mathbb{A}} F_{y_2}$: let $b \mathbin{\rotatebox[origin=c]{180}{$-\!\!*$}} c \in F_{x_2}$ and suppose $c \notin F_{y_2}$. Then $b \in f_w$. Using Zorn's lemma we can extend f_w to a prime filter F_w with these properties, and this suffices to show that the weak distributivity frame property is satisfied. □

The remaining structure for the dual adjunction is given in the same way as it was for **(B)BI**. We obtain functors by setting $Pr^{\mathrm{Bi(B)BI}}(f) = f^{-1}$ and $Com^{\mathrm{Bi(B)BI}}(g) = g^{-1}$ and we have natural transformations $\theta_{\mathbb{A}}(a) = \{F \mid a \in F\}$ and $\eta_{\mathcal{X}} = \{A \in Com^{\mathrm{Bi(B)BI}}(\mathcal{X}) \mid x \in A\}$. That θ respects the extra operations is verified by proving the existence of appropriate prime filters using Zorn's lemma and the properties of basic BiBBI algebras given in Proposition 6.9.

Theorem 6.16 (Soundness and Completeness for **Bi(B)BI** $+ \Sigma$)**.** *For any collection of **Bi(B)BI** properties Σ, $Pr^{\mathrm{Bi(B)BI}}$, $Com^{\mathrm{Bi(B)BI}}$, θ and η form a dual adjunction of categories between* $\mathrm{Bi(B)BIAlg}_{\Sigma}$ *and* $\mathrm{Bi(B)BIFr}_{\Sigma}$. □

Frame:	$\dfrac{\{p\}c\{q\}}{\{p * r\}c\{q * r\}}$	Concurrency:	$\dfrac{\{p_1\}c_1\{q_1\} \qquad \{p_2\}c_2\{q_2\}}{\{p_1 * p_2\}c_1 \parallel c_2\{q_1 * q_2\}}$
Skip:	$\dfrac{}{\{p\}\mathrm{skip}\{p\}}$	Seq:	$\dfrac{\{p\}c_1\{q\} \qquad \{q\}c_2\{r\}}{\{p\}c_1; c_2\{r\}}$
NonDet:	$\dfrac{\{p\}c_1\{q\} \qquad \{p\}c_2\{q\}}{\{p\}c_1 + c_2\{q\}}$	Iterate:	$\dfrac{\{p\}c\{p\}}{\{p\}\mathrm{Iterate}(c)\{p\}}$
Disjunction:	$\dfrac{\{p_i\}c\{q\},\ \text{all } i \in I}{\{\bigvee_{i\in I} p\}c\{q\}}$	Consequence:	$\dfrac{p \leq p' \qquad \{p\}c\{q\} \qquad q \leq q'}{\{p'\}c\{q'\}}$

Figure 10: Rules for $\mathbf{ASL}^{--}$.

We note that the completeness result for the logics $\mathbf{BiBI} + \Sigma$ is new. Indeed, it would not be possible to adapt the argument Brotherston & Villard give for **BiBBI** [14] directly as it relies on a translation into a Sahqlvist-axiomatized modal logic that uses Boolean negation in an essential way. There, the weak distributivity property is particularly difficult to deal with, requiring translation into a significantly more complicated frame property that is equivalent in the auxillary modal logic. In contrast, our proof is direct, and — with the groundwork of **(B)BI** duality done — efficient.

6.3. **Concurrent Kleene Algebra and Concurrent Separation Logic.** We conclude with a tentative application of our framework to Concurrent Separation Logic (**CSL**) [9, 58]. Without an account of the semantics of the programming language it is not immediately obvious how our duality theoretic approach can be extended to **CSL**, which requires strictly more structure than just the heap model of **FOBBI**. In any case, algebraic models of a basic version of **CSL**, $\mathbf{ASL}^{--}$, have been given that connect the logic to concurrent Kleene algebra [59]. The proof rules for $\mathbf{ASL}^{--}$ are given in Figure 10.

Definition 6.17 (Concurrent Kleene Algebra (cf. [59]))**.**

(1) A *concurrent monoid* $(M, \leq, \parallel, ;, \mathrm{skip})$ is a partial order $(M, \leq)$, together with two monoids $(M, \parallel, \mathrm{skip})$ (with $\parallel$ commutative) and $(M, ;, \mathrm{skip})$ satisfying the *exchange law*
$$(p \parallel r); (q \parallel s) \leq (p; q) \parallel (r; s).$$
It is *complete* if $(M, \leq)$ is a complete lattice.

(2) A *concurrent Kleene algebra* (CKA) is a complete concurrent monoid where $\parallel$ and ; preserve joins in both arguments.

(3) A *weak CKA* is a complete concurrent monoid together with a subset $A \subseteq M$ (the assertions of the algebra) such that i) $\mathrm{skip} \in A$; ii) A is closed under $\parallel$ and all joins; iii) $\parallel$ restricted to A preserves all joins in both arguments; iv) for each $a \in A$, $a; (-) : M \to M$ preserves all joins; and v) for each $m \in M$, $(-); m : A \to M$ preserves all joins.

(4) A CKA or weak CKA is *Boolean* if the underlying lattice is a Boolean algebra and intuitionistic if the underlying lattice is a Heyting algebra. □

O'Hearn et al. show that $\mathbf{ASL}^{--}$ is sound and complete for weak CKAs when Hoare triples $\{p\}c\{q\}$ are interpreted as inequalities $p; c \leq q$ (where $p, q \in A$ and $c \in M$) and $*$ is interpreted as $\|$ restricted to A. This is achieved via the construction of a predicate transformer model over $\mathbf{ASL}^{--}$ propositions. They also show that a trace model of $\mathbf{ASL}^{--}$ generates a Boolean CKA.

Elsewhere, O'Hearn [57] suggests that the structures involved could be used as inspiration for a bunched logic extending **BBI**. We define such a logic and call it *Concurrent Kleene BI* or **CKBI**. We leave the apparent intuitionistic variant extending **BI** to another occasion. The syntax and Hilbert-style proof theory of the logic is specifed by the following CKA-like algebra.

Definition 6.18 (CKBI Algebra)**.** A *CKBI algebra* is an algebra

$$\mathbb{A} = (A, \wedge, \vee, \neg, \top, \bot, \|, -\|, ;, -;, ;-, \mathrm{skip})$$

such that $(A, \wedge, \vee, \neg, \top, \bot, \|, -\|, \mathrm{skip})$ is a BBI algebra, $(A, ;, \mathrm{skip})$ is a monoid, $\|$ and ; satisfy the exchange law and for all $a, b, c \in A$, $a; b \leq c$ iff $a \leq b -; c$ iff $b \leq a ;- c$. □

A CKBI algebra is essentially a Boolean CKA extended with the multiplicative implications associated with $\|$ and ;. The properties of Proposition 3.2 ensure that $\|$ and ; are join preserving, and the properties of a BBI algebra make $(A, \|, \mathrm{skip})$ a commutative monoid. One difference is we do not require CKBI algebras to be complete. This is mainly for the sake of uniformity, though we note that — by the apparent representation theorem for CKBI algebras — every CKBI algebra can be embedded in a complete CKBI algebra.

Kripke models for **CKBI** are given on extended **BBI** frames that satisfy a first-order correspondent to the exchange law, enforcing a relationship between the parallel and sequential compositions.

Definition 6.19 (CKBI Frame)**.** A *CKBI frame* $\mathcal{X}$ is a tuple $\mathcal{X} = (X, \circ, \triangleright, E)$ such that
(1) $(X, \circ, E)$ is a BBI frame,
(2) $(X, \triangleright, E)$ is a non-commutative BBI frame; that is, it satisfies

(Assoc) $\exists t(t \in x \circ y \wedge w \in t \circ z) \leftrightarrow \exists s(s \in y \circ z \wedge w \in x \circ s)$
(Contr) $x \in (y \circ e) \cup (e \circ y) \wedge e \in E \to y = x$; (Weak) $\exists e \in E(x \in (x \circ e) \cap (e \circ x))$, *and*

(3) $\circ$ and $\triangleright$ satisfy

(Ex) $wy \in w \circ y \wedge xz \in x \circ z \wedge t \in wy \triangleright xz \to \exists wx, yz(wx \in w \triangleright x \wedge yz \in y \triangleright z \wedge t \in wx \circ yz)$.

□

The traces model of $\mathbf{ASL}^{--}$ can be seen as (the complex algebra of) a CKBI frame, where $\circ$ is interleaving, $\triangleright$ is concatenation and E is the singleton set containing the empty trace. Another example is given by pomsets [41], with $\circ$ given by the parallel pomset composition, $\triangleright$ the series pomset composition, and E the singleton set containing the empty pomset.

The semantics of **CKBI** on CKBI frames straightforwardly extends that of **BBI** by those in Figure 11, with the clauses for ; , $-$; and ;$-$ following those for **LGL**'s multiplicatives.

It is straightforward to adapt the results of Section 4 to these structures. The main additional verification is the correspondence between the frame condition (Ex) and the exchange law. Let $\mathbb{A}$ be a CKBI algebra. Then the prime filter frame of $\mathbb{A}$, $Pr^{\mathrm{CKBI}}(\mathbb{A})$, is given by extending the prime filter frame of the underlying BBI algebra with the operation $\triangleright_{\mathbb{A}}$, defined $F \triangleright_{\mathbb{A}} F' = \{F'' \mid \forall a \in F, \forall b \in F' : a; b \in F''\}$. In the other direction, the complex

$$x \vDash \phi\,;\psi \text{ iff there exists } y,z \text{ such that } x \in y \triangleright z,\ y \vDash \phi \text{ and } z \vDash \psi$$
$$x \vDash \phi{-;}\psi \text{ iff for all } y,z,\quad z \in x \triangleright y \text{ and } y \vDash \phi \text{ implies } z \vDash \psi$$
$$x \vDash \phi{;-}\psi \text{ iff for all } y,z,\quad z \in y \triangleright z \text{ and } y \vDash \phi \text{ implies } z \vDash \psi$$

Figure 11: Satisfaction for **CKBI**.

algebra of a CKBI frame $\mathcal{X}$, $Com^{\text{CKBI}}(\mathcal{X})$, is given by extending the complex algebra of the underlying BBI frame with the operation $A\,;_{\mathcal{X}}\,B = \{z \mid \exists x \in A, y \in B(z \in x \triangleright y)\}$ and its associated adjoints.

Lemma 6.20.

(1) *Given a CKBI algebra $\mathbb{A}$, the prime filter frame $Pr^{CKBI}(\mathbb{A})$ is a CKBI frame.*
(2) *Given a CKBI frame $\mathcal{X}$, the complex algebra $Com^{CKBI}(\mathcal{X})$ is a CKBI algebra.*

Proof. We concentrate on the correspondence between the exchange law and (Ex).

(1) Suppose we have prime filters of $\mathbb{A}$ satisfying $F_{wy} \in F_w \circ_{\mathbb{A}} F_y$, $F_{xz} \in F_x \circ_{\mathbb{A}} F_z$ and $F_t \in F_{wy} \triangleright_{\mathbb{A}} F_{xz}$. Consider the sets $f_{wx} = \{c \mid \exists a \in F_w, b \in F_x(a;b \leq c)\}$ and $f_{yz} = \{c \mid \exists a \in F_y, b \in F_z(a;b \leq c)\}$. Both sets are obviously upwards-closed and closed under meets, as order preservation of ; gives that $a;b \leq c$ and $a';b' \leq c'$ implies $(a \wedge a');(b \wedge b') \leq c \wedge c'$. Hence f_{wx} and f_{yz} are filters. They are also proper: suppose $\bot \in f_{wx}$. Then there exists $a \in F_w$ and $b \in F_x$ such that $a;b = \bot$. Let $c \in F_y$ and $d \in F_z$ be arbitrary. By assumption we have $a \parallel c \in F_{wy}$, $b \parallel d \in F_{xz}$ and so $(a \parallel c);(b \parallel d) \in F_t$. By the exchange law and upwards closure of prime filters, $(a;b) \parallel (c;d) \in F_t$: but $a;b = \bot$, so $(a;b) \parallel (c;d) = \bot \in F_t$, a contradiction.
   Clearly $f_{wx} \in F_w \triangleright F_x$ and $f_{yz} \in F_y \triangleright F_z$. Further, $F_t \in f_{wx} \circ_{\mathbb{A}} f_{yz}$: let $c \geq a;b$ and $c' \geq a';b'$ for $a \in F_w$, $b \in F_x$, $a' \in F_y$ and $b' \in F_z$. By order preservation of $\parallel$ and the exchange law, $(a \parallel a');(b \parallel b') \leq (a;b) \parallel (a';b') \leq c \parallel c'$. Since $(a \parallel a');(b \parallel b') \in F_t$, by upwards closure it follows that $c \parallel c' \in F_t$. We can thus use Zorn's lemma to obtain prime filters F_{wx} and F_{yz} that extend f_{wx} and f_{yz} and satisfy these properties.
(2) Suppose $t \in (A \parallel_{\mathcal{X}} C);_{\mathcal{X}}(B \parallel_{\mathcal{X}} D)$. Then there exist w, x, y, z, wy, xz such that $wy \in w \circ y$, $xz \in x \circ z$ and $t \in wy \triangleright xz$. The frame property (Ex) then ensures the existence of witnesses to the fact that $t \in (A;_{\mathcal{X}} B) \parallel_{\mathcal{X}} (C;_{\mathcal{X}} D)$. □

Once again the dual adjunction for **BBI** adapts straightforwardly to this extension by making Pr^{CKBI} and Com^{CKBI} functors in the usual fashion and considering the natural transformations θ and η.

Theorem 6.21 (Soundness and Completeness of **CKBI**)**.** *The functors Pr^{CKBI} and Com^{CKBI} and natural transformations θ and η form a dual adjunction between the categories* CKBIAlg *and* CKBIFr. □

We remain agnostic about the extent to which **CKBI** can be used to reason about **CSL**. The fact that it essentially supplies a Kripke semantics formulation of CKAs suggests that it may have uses as a logic for reasoning about concurrency more generally. We defer a thorough investigation of these ideas to another occasion.

7. Conclusions and Further Work

We have given a systematic treatment of Stone-type duality for the structures that interpret bunched logics, starting with the weakest systems, recovering the familiar **BI** and **BBI**, and concluding with both the classical and intutionistic variants of Separation Logic. Our results encompass all the known existing algebraic approaches to Separation Logic and prove them sound with respect to the standard store-heap semantics. As corollaries, we uniformly recover soundness and completeness theorems for the systems we consider. These results are extended to the bunched logics with additional multiplicatives corresponding to negation, disjunction and falsum — **DMBI**, **CBI** and the full range of sub-classical bunched logics — as well as **CKBI**, a new logic inspired by the algebraic structures that interpret (a basic version of) Concurrent Separation Logic. It is straightforward to use similar methods — in combination with the analogous results from the modal logic literature — to produce the same results for modal extensions of bunched logics [26, 27, 38] . We believe this treatment will simplify completeness arguments for future bunched logics by providing a modular framework within which existing results can be extended. This is demonstrated with our results on **DMBI**, **BiBI** and **CKBI**. More generally, the notion of indexed frame and its associated completeness argument can easily be adapted for a wide range of non-classical predicate logics.

We identify three areas of interest for further work. First, extendeding our approach to account for the operational semantics of program execution given by Hoare triples. As a consequence, we aim to interpret computational approaches to the Frame Rule such as bi-abduction [15] within our semantics and investigate if algebraic or topological methods can be brought to bear on these important aspects for implementations of Separation Logic. We believe the evident extension of our framework with Brink & Rewitzky's [8] duality-theoretic approach to Hoare logic could facilitate this. Such a set up should straightforwardly adapt to non-standard Separation Logics tailored for specific tasks (of which there are now many) which all essentially instantiate an indexed (B)BI frame structure. A related approach would be to use the dual adjunctions of the present work to generate state-and effect triangles for Separation Logic, which would enable us to investigate healthiness for pointer manipulating programs [42].

A second area of investigation concerns coalgebraic generalizations of Separation Logic. Previous work has given a sound and complete coalgebraic semantics for **(B)BI** [28] in line with coalgebraic generalizations of modal logic, but we are interested in another direction: the use of coalgbera as a mathematical foundation for transition systems. Separation Logic has shown the power of resource semantics for modelling real world phenomena when extended with suitable dynamics. Another approach utilizing process algebras generated by resource semantics has shown one way in which this idea can be applied to a more general class of distributed systems [2, 21]. We believe an analysis of coalgebras definable over the category of (B)BI frames would provide a general mathematical foundation for both these instances and more. A well developed line of work in coalgebraic logic has produced general machinery for generating sound and complete coalgebraic logics from dual adjunctions under favourable conditions [45, 50]. We thus believe such an analysis, combined with the dual adjunctions given in the present work, would yield a powerful framework for modelling, specifying and reasoning about resource-sensitive transition systems, with obvious wide applicability.

A third line of research would be the expansion of the results here to give a general treatment of categorical structures for non-classical predicate logics. The results given on

(B)BI hyperdoctrines straightforwardly apply to any hyperdoctrine with target algebras that have a duality theorem and include the Heyting or Boolean connectives. To what extent can this treatment generalize existing semantic approaches to non-classical predicate logics that fit this criteria? For example, the various categorical semantics given for predicate modal logic on sheaves [4], metaframes [63] and modal hyperdoctrines [7]. Our results could also be generalized in two ways. First, for hyperdoctrines with weaker-than-Heyting target algebras and their corresponding dual indexed frames, allowing us to encompass predicate substructural logics, predicate relevant logics and predicate positive logics. Second, to more exotic notions of quantification. As one example, further extending our framework to encompass the breadth of the bunched logic literature would require an account of multiplicative quantification, an area which has only partially been explored algebraically. Collinson et al. [22] define category theoretic structures that combine hyperdoctrines with monoidal categories to give semantics to a bunched polymorphic lambda calculus. There, the right adjoint of $\mathbb{P}(\pi_{\Gamma,X})$ must satisfy a compatibility condition with the monoidal structure in order to interpret multiplicative universal quantification in the calculus, and — with some subtle technical tweaks and additional structure — it is possible to give left adjoints that can interpret multiplicative existential quantification. Is there a unification of this approach with the present work that adapts these ideas to the more general logical setting? We believe our framework provides the mathematical foundation to explore these ideas.

References

[1] A.R. Anderson, N. Belnap, and J.M. Dunn. *Entailment. The Logic of Relevance and Necessity. Vol. II.* Princeton University Press, Princeton, NJ, 1992. With contributions by Kit Fine, Alasdair Urquhart et al, Includes a bibliography of entailment by Robert G. Wolf.

[2] G. Anderson and D. Pym. A Calculus and Logic of Bunched Resources and Processes. *Theoret. Comput. Sci.*, 614:63–96, 2016.

[3] C. Areces and B. ten Cate. Hybrid logics. In *Handbook of Modal Logic*, Studies in Logic and Practical Reasoning Vol 3, P. Blackburn, J. van Benthem, F. Wolter eds. 821–868, Elsevier, 2007.

[4] S. Awodey and K. Kishida. Topological Completeness of First-Order Modal Logic. *Advances in Modal Logic*, 9:1–17, 2012.

[5] B. Biering, L. Birkedal, and N. Torp-Smith. BI Hyperdoctrines and Higher-order Separation Logic. In *Programming Languages and Systems: 14th European Symposium on Programming*, ESOP 2005, Edinburgh, Scotland, Springer-Verlag, 233–247, 2005.

[6] K. Bimbó and J.M. Dunn. *Generalized Galois Logics. Relational Semantics of Nonclassical Logical Calculi.* CSLI Lecture Notes Vol 188, CSLI Publications, Stanford, December 2008.

[7] T. Braüner and S. Ghilardi. First-Order Modal Logic. In *Handbook of Modal Logic*, Studies in Logic and Practical Reasoning Vol 3, P. Blackburn, J. van Benthem, F. Wolter eds, Elsevier, 549–620, 2007.

[8] C. Brink and I. Rewitzky. *A Paradigm for Program Semantics: Power Structures and Duality*, Studies in Logic, Language and Information. CSLI Publications, Stanford, CA, 2001.

[9] S. Brookes. A semantics for concurrent separation logic. *Theor. Comput. Sci.*, 375(13): 227–270, 2007.

[10] J. Brotherston. Bunched Logics Displayed. *Studia Logica* 100(6):1223-1254, 2012.

[11] J. Brotherston and C. Calcagno. Classical BI: Its semantics and proof theory. *Log. Methods Comput. Sci.*, 6 (3)1-42. 2010. `http:/doi.org/10.2168/LMCS-6(3:3)2010`

[12] J. Brotherston and M. Kanovich. Undecidability of Propositional Separation Logic and Its Neighbours. *J. ACM*, 61(2):`https:doi.org/10.1145/2542667`, 2014.

[13] J. Brotherston and J. Villard. Parametric Completeness for Separation Theories. *POPL '14: The 41st Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, San Diego, CA, USA, ACM, 453–464, 2014. `https://doi.org/10.1145/2535838.2535844`

[14] J. Brotherston and J. Villard. Sub-Classical Boolean Bunched Logics and the Meaning of Par. In *24th EACSL Annual Conference on Computer Science Logic*, CSL 2015, Berlin, Germany, LIPlcs 41, Dagstuhl, 325–342, 2015.

[15] C. Calcagno, D. Distefano, P. O'Hearn, and H. Yang. Compositional Shape Analysis by Means of Bi-abduction. *J. ACM*, 58(6): `https://doi.org/10.1145/2049697.2049700`, 2011.

[16] C. Calcagno, P. Gardner, and U. Zarfaty. Context logic as modal logic: Completeness and parametric inexpressivity. In *POPL '07: The 34th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, Nice, France, ACM, 123–134, 2007.

[17] C. Calcagno, P. O'Hearn, and H. Yang. Local Action and Abstract Separation Logic. In *22nd Annual IEEE Symposium on Logic In Computer Science*, LICS 2007, Wroclaw, Poland, IEEE, 366–378, 2007.

[18] Q. Cao, S. Cuellar, and A. Appel. Bringing Order to the Separation Logic Jungle In *15th Asian Symposium on Programming Languages and Systems*, APLAS 2017, Suzhou, China, LNCS, to appear.

[19] M. Collinson, K. McDonald, and D. Pym. A Substructural Logic for Layered Graphs. *J. Logic Comput.*, 24(4):953–988, 2014. `https://doi.org/10.1093/logcom/exu002`

[20] M. Collinson, K. McDonald, and D. Pym. Layered Graph Logic as an Assertion Language for Access Control Policy Models. *J. Logic Comput.* , 27(1):4180, 2017. `https://doi.org/10.1093/logcom/exv020`

[21] M. Collinson and D. Pym. Algebra and logic for resource-based systems modelling. *Math. Structures Comput. Sci.* , 19:959–1027, 2009. `https://doi.org/10.1017/S0960129509990077`

[22] M. Collinson, D. Pym and E. Robinson. Bunched Polymorphism. *Math. Structures Comput. Sci*, 18(6):1091–1132, 2008. `http://doi.org/10.1017/S0960129508007159`

[23] W. Conradie and C. Robinson. On Sahlqvist theory for hybrid logics. *J. Logic Comput.* 27(3):867-900, 2017. `https://doi.org/10.1093/logcom/exv045`

[24] D. Coumans. Duality for first-order logic. `http://www.math.ru.nl/~coumans/talkAC.pdf`. Accessed 15 Sept 2017.

[25] D. Coumans. Generalising Canonical Extension to the Categorical Setting. *Ann. Pure Appl. Logic*, 163(12):1940–1961, 2012.

[26] J.-R. Courtault and D. Galmiche. A Modal BI Logic for Dynamic Resource Properties. In *Logical Foundations of Computer Science International Symposium*, LFCS 2013, San Diego, CA, USA, Springer Berlin Heidelberg, 134-138, 2013.

[27] J.-R. Courtault, D. Galmiche, and D. Pym. A Logic of Separating Modalities. *Theoret. Comput. Sci.*, 637:30–58, 2016.

[28] F. Dahlqvist and D. Pym. Completeness via Canonicity for Distributive Substructural Logics: A Coalgebraic Perspective. In *Relational and Algebraic Methods in Computer Science: 15th International Conference*, RAMICS 2015, Braga, Portugal, Springer, 119–135, 2015.

[29] H.-H. Dang and P. Höfner and B. Möller. Algebraic Separation Logic. *J. Log. Algebr. Program.* , 80(6):221–247, 2011.

[30] S. Docherty and D. Pym. A Stone-type Duality Theorem for Separation Logic via its Underlying Bunched Logics. In *Mathematical Foundations of Programming Semantics, 9th International Conference*, MFPS 33, Llubljana, Slovenia LNCS, 2017.

[31] S. Docherty and D. Pym. A Stone-type Duality Theorem for Separation Logic via its Underlying Bunched Logics: Research Note. Research note RN/17/06, Department of Computer Science, UCL, `http://www.cs.ucl.ac.uk/fileadmin/UCL-CS/research/Research_Notes/RN_17_06.pdf`, 2017

[32] S. Docherty and D. Pym. Intuitionistic Layered Graph Logic. In *Automated Reasoning: 8th International Joint Conference*, IJCAR 2016, Coimbra, Portugal, LNAI 9706:469–486, 2016.

[33] R. Dockins, A. Hobor, and A.W. Appel. A Fresh Look at Separation Algebras and Share Accounting. In *Proc. of the 7th Asian Symposium on Programming Languages and Systems*, APLAS '09, 161–177, Berlin, Heidelberg, 2009. Springer-Verlag.

[34] B. Dongol, V. Gomes, and G. Struth. A Program Construction and Verification Tool for Separation Logic. In *Mathematics of Program Construction: 12th International Conference*, MPC 2015, Knigswinter, Germany, Springer, 137–158, 2015.

[35] J. M. Dunn. Gaggle Theory: An Abstraction of Galois Connections and Residuation with Applications to Negation, Implication, and Various Logical Operations. *Logics in AI: European Workshop JELIA '90*, Springer-Verlarg, 31–51, 1990.

[36] J. M. Dunn and G. Hardegree. *Algebraic Methods in Philosophical Logic* Oxford University Press, 2001.

[37] L. Esakia. Topological Kripke models. *Soviet Math. Dokl.* 15, 147–15, 1974.

[38] D. Galmiche, P. Kimmel, and D. Pym. A Substructural Epistemic Resource Logic. In *Logic and Its Applications: 7th Indian Conference*, ICLA '17, Kanpur, India, Springer, 106–122, 2017.

[39] D. Galmiche and D. Larchey-Wendling. Expressivity Properties of Boolean BI Through Relational Models. In *FSTTCS 2006: Foundations of Software Technology and Theoretical Computer Science*, Kolkata, India, Springer Berlin Heidelberg, 357–368, 2006.

[40] D. Galmiche, D. Méry, and D. Pym. The Semantics of **BI** and Resource Tableaux. *Math. Structures Comput. Sci*, 15:1033–1088, 2005.

[41] J. Gischer. The equational theory of pomsets. *Theoret. Comput. Sci.*, 61(2-3):199 – 224, 1988. `https://doi.org/10.1016/0304-3975(88)90124-7`

[42] W. Hino, H. Kobayashi, I. Hasuo and B. Jacobs. Healthiness from Duality. In *Thirty-First Annual ACM/IEEE Symposium on Logic In Computer Science*, LICS 2016, New York, USA, ACM/IEEE, 682–691, 2016.

[43] R. Goldblatt. Varieties of Complex Algebras. *Ann. Pure Appl. Logic*, 44(3):173-242, 1989.

[44] S. Ishtiaq and P. O'Hearn. **BI** as an Assertion Language for Mutable Data Structures. In *POPL '01: 28th ACM-SIGPLAN Symposium on Principles of Programming Languages*, London, UK, ACM, 14–26. ACM, 2001.

[45] B. Jacobs and A. Sokolova. Exemplaric expressivity of modal logics. *J. Logic Comput.* , 20(5):1041 - 1068, 2010.

[46] P. Jipsen and C. Tsinakis. A survey of residuated lattices. In *Ordered Algebraic Structures*, Developments In Mathematics, Springer, 19-56, 2002.

[47] P. Johnstone. *Stone Spaces*. Cambridge University Press, 1986.

[48] B. Jonsson and A. Tarski. Boolean algebras with operators. Part I. *Amer. J. Math.*, 73(4):891–939, 1951.

[49] R. Jung, R. Krebbers, J.-H. Jourdan, A. Bizjak, L. Birkedal and D. Dreyer. Iris from the ground up: A modular foundation for higher-order concurrent separation logic. under consideration for publication in *J. Functional Programming*, 2017.

[50] C. Kupke and D. Pattinson. Coalgebraic semantics of modal logics: An overview. *Theoret. Comput. Sci.* 412(38):5070-5094, 2011.

[51] J. Lambek. On the calculus of syntactic types. In *Studies of Language and its Mathematical Aspects*, Proc. of the 12th Symp. Appl. Math., 166–178, 1961.

[52] J. Lambek. From categorical grammar to bilinear logic. In *Substructural Logics*, P. Schroeder-Heister and K. Došen eds, Oxford University Press, 207–237, 1993.

[53] D. Larchey-Wendling. The Formal Strong Completeness of Partial Monoidal Boolean BI. *J. Log. Comput.*, 26(2):605–640, 2016.

[54] D. Larchey-Wendling and D. Galmiche. Looking at Separation Algebras with Boolean BI-eyes. In *Theoretical Computer Science: 8th IFIP TC 1/WG 2.2 International Conference*, TCS 2014, Rome, Italy, Springer Berlin Heidelberg, 326–340, 2014.

[55] H. Nakano. A Modality for Recursion. In *Fifteenth Annual IEEE Symposium on Logic in Computer Science*, LICS 2000, Santa Barbara, California, IEEE 255–266, 2000.

[56] P. O'Hearn. A Primer on Separation Logic. *Software Safety and Security; Tools for Analysis and Verification. NATO Science for Peace and Security Series*, 33:286–318, 2012.

[57] P. O'Hearn. Algebra, Logic, Locality, Concurrency. `http://www0.cs.ucl.ac.uk/staff/p.ohearn/Talks/APLAS-CPP-2011.pdf`. Accessed 15 Sept 2017.

[58] P. O'Hearn. Resources, concurrency, and local reasoning. *Theoret. Comput. Sci.*, 375(1-3): 271–307, 2007.

[59] P. O'Hearn, R. Petersen, J. Villard , A. Hussain. On the relation between Concurrent Separation Logic and Concurrent Kleene Algebra. *J. Log. Algebr. Methods Program.*, 84(3): 285–302, 2015.

[60] P. O'Hearn and D. Pym. The Logic of Bunched Implications. *Bull. Symb. Log.*, 5(2):215–244, June 1999.

[61] A. Pitts. Categorical Logic. In *Handbook of Logic in Computer Science, Volume 5*, S. Abramsky and D. Gabbay and T. Maibaum, eds, Oxford University Press, 39–128, 2000.

[62] J. Reynolds. Separation Logic: A Logic for Shared Mutable Data Structures. In *Seventeenth Annual IEEE Symposium on Logic In Computer Science*, LICS 2002, Copenhagen, Denmark, IEEE, 55–74, 2002.

[63] H. Shirasu. Duality in Superintuitionistic and Modal Predicate Logics. *Advances in Modal Logic*, 1:223–236, 1998.

[64] M. H. Stone. The Theory of Representations of Boolean Algebras. *Trans. Amer. Math. Soc.* 40, 37–111. 1936.

[65] A. Troelstra and H. Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, 1996.

[66] J. Van Benthem. Correspondence Theory. In *Handbook of Philosophical Logic* Vol 3, D. M. Gabbay and F Guenthner eds, Springer Netherlands, 67–247, 2001.

[67] H. Yang and P. O'Hearn. A Semantic Basis for Local Reasoning. In *Proc. Foundations of Software Science and Computation Structures 5th International Conference*, FOSSACS 2002, Grenoble, France, Springer, 402–416, 2002. https://doi.org/10.1007/3-540-45931-6_28